Journal of the Iranian Mathematical Society
ISSN (on-line): 2717-1612
J. Iranian Math. Soc.

A GENERALIZATION OF POSNER'S THEOREM ON GENERALIZED DERIVATIONS IN RINGS

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Abstract. In this paper, we generalize the Posner's theorem on generalized derivations in rings as follows: Let $\mathscr{A}$ be an arbitrary ring, $\mathscr{I}$ a non-zero ideal, $\mathscr{T}$ is a prime ideal of $\mathscr{A}$ such that $\mathscr{T} \subset \mathscr{I}$, and ψ be a non-zero generalized derivation associated with a non-zero derivation ρ of $\mathscr{A}$. If one of the following conditions is satisfied: (i) $[\psi(x), x] \in \mathscr{T}$, (ii) $[[\psi(x), x], y] \in \mathscr{T}$, (iii) $\overline{[\psi(x), x]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T})$ and (iv) $\overline{[[\psi(x), x], y]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T})$ $\forall\ x, y \in \mathscr{I}$, then $\rho(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative. At the example, it is given that the hypothesis of the theorems are necessary.

1. Introduction

The whole paper, the symbol $[x, y]$, where $x, y \in \mathscr{A}$, stands for the commutator $xy - yx$. An ideal $\mathscr{T}$ is said to be a prime ideal of $\mathscr{A}$ if $\mathscr{T} \neq \mathscr{A}$ and for any $x, y \in \mathscr{A}$, whenever $x\mathscr{A}y \subseteq \mathscr{T}$ implies $x \in \mathscr{T}$ or $y \in \mathscr{T}$ and $\mathscr{A}$ is a prime ring if $\mathscr{T} = 0$, and $\mathscr{T}$ is a semiprime ideal if $\mathscr{T} \neq \mathscr{A}$ and for any $x \in \mathscr{A}$, $x\mathscr{A}x \subseteq \mathscr{T}$ implies $x \in \mathscr{T}$ and $\mathscr{A}$ is a semiprime ring if $\mathscr{T} = 0$. A ring $\mathscr{A}$ is said to be 2-torsion free if $2x = 0$, $x \in \mathscr{A}$ implies $x = 0$. A map $f : \mathscr{S} \to \mathscr{A}$ is called a $\mathscr{D}$-commuting map on $\mathscr{S}$ if $[f(x), x] \in \mathscr{D}$ $\forall\ x \in \mathscr{S}$ and some $\mathscr{D} \subseteq \mathscr{A}$. In particular, if $\mathscr{D} = \{0\}$, then f is called a commuting map on $\mathscr{S}$ if $[f(x), x] = 0$. Note that every commuting map is a $\mathscr{D}$-commuting map (put $\{0\} = \mathscr{D}$). But the converse is not true in general. Take $\mathscr{D}$ some a set of $\mathscr{A}$ has no zero such that $[f(x), x] \in \mathscr{D}$, then f is a $\mathscr{D}$-commuting map but it is not a commuting map. An additive mapping $\rho : \mathscr{A} \to \mathscr{A}$ is

Communicated by Saeid Azam

MSC(2020): Primary: 16N60; Secondary: 16U80, 16W25.

Keywords: Prime ideal, commutativity, derivation, generalized derivations.

Received: 25 March 2022, Accepted: 16 November 2022.

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DOI: https://dx.doi.org/10.30504/JIMS.2022.335190.1059

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called a derivation if $\rho(xy) = \rho(x)y + x\rho(y)$, for all $x, y \in \mathscr{A}$. An additive mapping $\psi : \mathscr{A} \to \mathscr{A}$ is called a generalized derivation if there exists a derivation $\rho : \mathscr{A} \to \mathscr{A}$ such that $\psi(xy) = \psi(x)y + x\rho(y)$ for all $x, y \in \mathscr{A}$.

The history of commuting and centralizing were studied by Divinsky in [3]. He proved that a simple Artinian ring is commutative if it has a commuting nontrivial automorphism. The commutativity of prime rings with derivation was initiated by Posner in [8]. He showed that if a prime ring has a nontrivial derivation which is centralizing on the entire ring, then the ring must be commutative. In [2], Awtar considered centralizing derivations on Lie and Jordan ideals. Awtar showed that if there is centralizing derivation on the Lie ideal of characteristic not two or three pime rings, the ideal is contained in the center. In [5], Lee et al. proved the same theorem by removing the characteristic not three. Recently, many authors studied this subject for different mappings on prime and semiprime rings see [4, 6, 7, 9]. In [1], Almahdi et al. proved Posner's theorem given above as follows: if $\mathscr{T}$ is a prime ideal of a ring $\mathscr{A}$ and ρ a derivation of $\mathscr{A}$ such that $[[\rho(x), x], y] \in \mathscr{T}$ $\forall$ $x, y \in \mathscr{A}$, then $\rho(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is a commutative ring. In particular, if $\mathscr{A}$ is semiprime and ρ is a centralizing derivation of $\mathscr{A}$, then either $\mathscr{A}$ is commutative or there exists a minimal prime ideal $\mathscr{T}$ of $\mathscr{A}$ such that $\rho(\mathscr{A}) \subseteq \mathscr{T}$. As a consequence, for any semiprime ring with a centralizing derivation there exists at least a minimal prime ideal $\mathscr{T}$ such that $\rho(\mathscr{T}) \subseteq \mathscr{T}$. For more details, see [10–14].

In this paper, the theorem given above is proved for ring with generalized derivation.

2. The main result

Lemma 2.1. *Let $\mathscr{A}$ be a ring with $\mathscr{T}$ a semiprime ideal, $\mathscr{I}$ a non-zero ideal and $\mathscr{A}/\mathscr{T}$ a 2-torsion free. If $\psi : \mathscr{A} \to \mathscr{A}$ is additive mapping such that $\overline{[\psi(x), x]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T})$, then ψ is $\mathscr{T}$-commuting on $\mathscr{I}$.*

Proof. By the hypothesis, we get

$$\overline{[\psi(x), x]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T}), \tag{2.1}$$

for all $x \in \mathscr{I}$. By linearizing (2.1), we obtain that

$$\overline{[\psi(x), y] + [\psi(y), x]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T}),$$

for all $x, y \in \mathscr{I}$. Putting $y = x^2$ in the last relation, we have

$$\overline{[\psi(x), x^2] + [\psi(x^2), x]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T}).$$

By applying (2.1) in the last relation, we get

$$\overline{2[\psi(x), x]x + [\psi(x^2), x]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T}), \tag{2.2}$$

for all $x \in \mathscr{I}$. By applying (2.1), we get $\overline{[\psi(x^2), x^2]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T})$.

That is

$$\overline{[\psi(x^2), x]x + x[\psi(x^2), x]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T}), \tag{2.3}$$

DOI: https://dx.doi.org/10.30504/JIMS.2022.335190.1059

for all $x \in \mathscr{I}$. Now fix $x \in \mathscr{I}$ and let $\overline{z} = \overline{[\psi(x), x]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T})$, $\overline{u} = \overline{[\psi(x^2), x]}$. We must show that $z \in \mathscr{T}$. From (2.2), we obtain $\overline{[\psi(x), 2zx + u]} = (\overline{0})$. That is $[\psi(x), 2zx + u] \in \mathscr{T}$. Hence, we get

$$2z^2 + [\psi(x), u] \in \mathscr{T}, \tag{2.4}$$

for all $x \in \mathscr{I}$. By applying (2.3), we see that $\overline{[\psi(x), ux + xu]} = (\overline{0})$. That is

$$[\psi(x), ux + xu] \in \mathscr{T}$$

and so

$$[\psi(x), u]x + u[\psi(x), x] + [\psi(x), x]u + x[\psi(x), u] \in \mathscr{T}.$$

This implies that

$$[\psi(x), u]x + 2uz + x[\psi(x), u] \in \mathscr{T}.$$

By using (2.4) in the last relation, we have $-4z^2x + 2uz \in \mathscr{T}$ and so $uz - 2z^2x \in \mathscr{T}$. In the last relation, multiply left and right with $\psi(x)$, we have

$$\psi(x)(uz - 2z^2x) \in \mathscr{T} \text{ and } (uz - 2z^2x)\psi(x) \in \mathscr{T}.$$

When compared this two relation, we get

$$[\psi(x), uz - 2z^2x] \in \mathscr{T}.$$

That is

$$[\psi(x), uz] - [\psi(x), 2z^2x] \in \mathscr{T}.$$

We obtain that

$$z[\psi(x), u] - [\psi(x), 2z^2x] \in \mathscr{T}.$$

Multiplying (2.4) by z and using the last relation, we get $2z^3 + [\psi(x), 2z^2x] \in \mathscr{T}$ and so $2z^3 + 2z^2[\psi(x), x] \in \mathscr{T}$. This implies that $2z^3 + 2z^3 \in \mathscr{T}$. We conclude that $4z^3 \in \mathscr{T}$. Since $\mathscr{A}/\mathscr{T}$ is a 2-torsion free, $\mathscr{T}$ is a 2-torsion free ring. That is, $z^3 \in \mathscr{T}$. We see that $\overline{z^3} = (\overline{0})$. Since $\mathscr{T}$ is a semiprime ideal, $\mathscr{A}/\mathscr{T}$ is a semiprime ring. The center of a semiprime ring does not contain non-zero nilpotents. We see that $\overline{z} = (\overline{0})$. Thus, $z \in \mathscr{T}$. □

Lemma 2.2. *Let $\mathscr{I}$ be a non-zero ideal in a ring $\mathscr{A}$, $\mathscr{T}$ a prime ideal such that $\mathscr{T} \subset \mathscr{I}$ and ρ a derivation of $\mathscr{A}$. If $\rho(\mathscr{I}) \subseteq \mathscr{T}$, then $\rho(\mathscr{A}) \subseteq \mathscr{T}$.*

Proof. Assume that $\rho(\mathscr{I}) \subseteq \mathscr{T}$ and so $\rho(r)x = \rho(rx) - r\rho(x) \in \mathscr{T}$, for all $x \in \mathscr{I}$, $r \in \mathscr{A}$. That is $\rho(r)x \in \mathscr{T}$. We obtain that $\rho(r)\mathscr{A}x \subseteq \mathscr{T}$. But $\mathscr{T} \subsetneqq \mathscr{I}$. There exists $x_0 \in \mathscr{I}$ such that $x_0 \notin \mathscr{T}$. We see that $\rho(r)\mathscr{A}x_0 \subseteq \mathscr{T}$, for all $r \in \mathscr{A}$. Hence $\rho(r) \in \mathscr{T}$, for all $r \in \mathscr{A}$. Therefore $\rho(\mathscr{A}) \subseteq \mathscr{T}$. □

Lemma 2.3. *Let $\mathscr{A}$ be a ring, $\mathscr{I}$ a non-zero ideal, $\mathscr{T}$ a prime ideal of $\mathscr{A}$ such that $\mathscr{T} \subset \mathscr{I}$. If $[\mathscr{I}, \mathscr{I}] \subseteq \mathscr{T}$, then $\mathscr{A}/\mathscr{T}$ is commutative.*

DOI: https://dx.doi.org/10.30504/JIMS.2022.335190.1059

Proof. If $x \in \mathscr{I}$, then $\mathscr{I}_x(\mathscr{I}) = [x, \mathscr{I}] \subseteq \mathscr{T}$, since $[\mathscr{I}, \mathscr{I}] \subseteq \mathscr{T}$. Using $\mathscr{I}_x$ derivation, we get $\mathscr{I}_x(\mathscr{A}) \subseteq \mathscr{T}$ by Lemma 2.2. That is $[x, \mathscr{A}] \subseteq \mathscr{T}$, for all $x \in \mathscr{I}$. We obtain that$[\mathscr{I}, \mathscr{A}] \subseteq \mathscr{T}$. That is, $[\mathscr{A}, \mathscr{I}] \subseteq \mathscr{T}$. This implies that $\mathscr{I}_r(\mathscr{I}) = [r, \mathscr{I}] \subseteq \mathscr{T}$, for all $r \in \mathscr{A}$. By Lemma 2.2, we get $\mathscr{I}_r(\mathscr{A}) \subseteq \mathscr{T}$, for all $r \in \mathscr{A}$. That is $[\mathscr{A}, \mathscr{A}] \subseteq \mathscr{T}$. Therefore, we conclude that $\mathscr{A}/\mathscr{T}$ is commutative. □

Proof. Assume that $[x, y] \in \mathscr{T}$ for all $x, y \in \mathscr{I}$, and $\overline{\mathscr{A}} = \mathscr{A}/\mathscr{T}$. Then $\overline{[x, y]} = (\overline{0})$. That is $[\overline{x}, \overline{y}] = (\overline{0})$. By Lemma 2.3, we conclude that $\overline{\mathscr{A}}$ is a commutative. □

Lemma 2.4. *Let $\mathscr{A}$ be a ring, $\mathscr{I}$ a non-zero ideal, $\mathscr{T}$ a prime ideal of $\mathscr{A}$ such that $\mathscr{T} \subset \mathscr{I}$ and $x \in \mathscr{A}$. If $\overline{x} \in \mathscr{Z}(\mathscr{I}/\mathscr{T})$, then $\overline{x} \in \mathscr{Z}(\mathscr{A}/\mathscr{T})$.*

Proof. Let $x \in \mathscr{A}$. If $\overline{x} \in \mathscr{Z}(\mathscr{I}/\mathscr{T})$, then $[x, \mathscr{I}] \subseteq \mathscr{T}$. This implies that $[x, \mathscr{I}\mathscr{A}] \subseteq \mathscr{T}$. We obtain that $[x, \mathscr{I}]\mathscr{A} + \mathscr{I}[x, \mathscr{A}] \subseteq \mathscr{T}$. This implies that $\mathscr{I}[x, \mathscr{A}] \subseteq \mathscr{T}$, but $\mathscr{T} \subset \mathscr{I}$ and so $[x, \mathscr{A}] \subseteq \mathscr{T}$. Hence $[\overline{x}, \mathscr{A}/\mathscr{T}] = (\overline{0})$. Therefore, we get $\overline{x} \in \mathscr{Z}(\mathscr{A}/\mathscr{T})$. □

Theorem 2.5. *Let $\mathscr{A}$ be a ring, $\mathscr{I}$ a non-zero ideal, $\mathscr{T}$ a prime ideal of $\mathscr{A}$ such that $\mathscr{T} \subset \mathscr{I}$. If (ψ, ρ) is a non-zero generalized derivation of $\mathscr{A}$ and ψ is a $\mathscr{T}$-commuting on $\mathscr{I}$, then $\rho(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative.*

Proof. Assume that

$$[\psi(x), x] \in \mathscr{T}, \tag{2.5}$$

for all $x \in \mathscr{I}$. By linearizing (2.5), we have

$$[\psi(x), y] + [\psi(y), x] \in \mathscr{T}, \tag{2.6}$$

for all $x, y \in \mathscr{I}$. Replacing y by yx in (2.6) and applying (2.6) and (2.5), we get

$$[y\rho(x), x] \in \mathscr{T}, \tag{2.7}$$

for all $x, y \in \mathscr{I}$. Substituting ty for $y, t \in \mathscr{A}$ in (2.7) and using (2.7), we get $[t, x]y\rho(x) \in \mathscr{T}$. Since $\mathscr{T}$ is a prime ideal, we get $[t, x] \in \mathscr{T}$ or $\rho(x) \in \mathscr{T}$. Let $\mathscr{K} = \{x \in \mathscr{I} \mid [t, x] \in \mathscr{T}\}$ and $\mathscr{J} = \{x \in \mathscr{I} \mid \rho(x) \in \mathscr{T}\}$. A group cannot be union of its subgroups. If $\mathscr{K} = \mathscr{A}$, then $\mathscr{A}/\mathscr{T}$ is commutative by Lemma 2.3. If $\mathscr{J} = \mathscr{A}$, then $\rho(x) \in \mathscr{T}$, for all $x \in \mathscr{I}$. That is $\rho(\mathscr{I}) \subseteq \mathscr{T}$. By Lemma 2.2, we have $\rho(\mathscr{I}) \subseteq \mathscr{T}$. □

Corollary 2.6. *([1], Lemma 2.1) Let $\mathscr{A}$ be a ring, $\mathscr{I}$ an ideal, $\mathscr{T}$ a prime ideal of $\mathscr{A}$ such that $\mathscr{T} \subset \mathscr{I}$ and ρ a non-zero derivation. If $[\rho(x), x] \in \mathscr{T}$, for all $x \in \mathscr{I}$, then either $\rho(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative.*

Theorem 2.7. *Let $\mathscr{A}$ be a ring, $\mathscr{I}$ a non-zero ideal, $\mathscr{T}$ a prime ideal of $\mathscr{A}$ such that $\mathscr{T} \subset \mathscr{I}$. If (ψ, ρ) is a non-zero generalized derivation of $\mathscr{A}$ such that $\overline{[\psi(x), x]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T})$, for all $x \in \mathscr{I}$, then $\rho(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative.*

DOI: https://dx.doi.org/10.30504/JIMS.2022.335190.1059

Proof. Suppose that $char(\mathscr{A}/\mathscr{T}) \neq 2$ and $\overline{[\psi(x), x]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T})$. By Lemma 2.1, we see that $\overline{[\psi(x), x]} = (\bar{0})$. That is $[\psi(x), x] \in \mathscr{T}$. By Theorem 2.5, we get $\mathscr{A}/\mathscr{T}$ is commutative or $\rho(\mathscr{A}) \subseteq \mathscr{T}$.

Now, if $char(\mathscr{A}/\mathscr{T}) = 2$, then the operation $(-)$ will be written $(+)$. By applying $\overline{[\psi(x), x]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T})$, we get

$$[[\psi(x), x], r] \in \mathscr{T}, \tag{2.8}$$

for all $x \in \mathscr{I}$ and $r \in \mathscr{A}$. By linearizing (2.8), we get

$$[[\psi(x), y] + [\psi(y), x], r] \in \mathscr{T}, \tag{2.9}$$

for all $x, y \in \mathscr{I}$ and $r \in \mathscr{A}$. By applying (2.8) and (2.9) in equation (2.10), we have

$$\begin{aligned} &[xy + yx, \psi(x)] + [x^2, \psi(y)] \\ &= x([y, \psi(x)] + [x, \psi(y)]) + ([y, \psi(x)] + [x, \psi(y)])x + y[x, \psi(x)] + [x, \psi(x)]y \in \mathscr{T}, \end{aligned} \tag{2.10}$$

for all $x, y \in \mathscr{I}$. That is

$$[xy + yx, \psi(x)] + [x^2, \psi(y)] \in \mathscr{T}, \tag{2.11}$$

for all $x, y \in \mathscr{I}$. Replacing y by x in (2.11) and using $char(\mathscr{A}/\mathscr{T}) = 2$, we have

$$[x^2, \psi(x)] \in \mathscr{T}, \tag{2.12}$$

for all $x, y \in \mathscr{I}$. Writing yx instead of y in (2.11) and using (2.11), we obtain

$$(xy + yx)[x, \psi(x)] + y[x^2, \rho(x)] + [x^2, y]\rho(x) \in \mathscr{T}, \tag{2.13}$$

for all $x, y \in \mathscr{I}$. Replacing y by ry in (2.11), we see that

$$(xry + ryx)[x, \psi(x)] + ry[x^2, \rho(x)] + r[x^2, y]\rho(x) + [x^2, r]y\rho(x) \in \mathscr{T}, \tag{2.14}$$

for all $x, y \in \mathscr{I}$ and $r \in \mathscr{A}$. Multiplying the equation (2.13) on the left for r, we find that

$$(rxy + ryx)[x, \psi(x)] + ry[x^2, \rho(x)] + r[x^2, y]\rho(x) \in \mathscr{T}, \tag{2.15}$$

for all $x, y \in \mathscr{I}$ and $r \in \mathscr{A}$. Comparing (2.13) and (2.14), we conclude that

$$[x, r]y[x, \psi(x)] + [x^2, r]y\rho(x) \in \mathscr{T},$$

for all $x, y \in \mathscr{I}$ and $r \in \mathscr{A}$. Replacing r by $\psi(x)$ in (2.15) and using (2.12), we have $[x, \psi(x)]y[x, \psi(x)] \in \mathscr{T}$ and so $[x, \psi(x)] \in \mathscr{T}$. By Theorem 2.5, we get $\rho(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative. □

Corollary 2.8. *Let $\mathscr{A}$ be a ring, $\mathscr{I}$ a non-zero ideal, $\mathscr{T}$ a prime ideal of $\mathscr{A}$ such that $\mathscr{T} \subset \mathscr{I}$. If ρ is a non-zero derivation such that $\overline{[\rho(x), x]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T})$, for all $x \in \mathscr{I}$, then $\rho(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative.*

Theorem 2.9. *Let $\mathscr{A}$ be a ring and $\mathscr{I}$ a non-zero ideal of $\mathscr{A}$, $\mathscr{T}$ a prime ideal of $\mathscr{A}$ such that $\mathscr{T} \subset \mathscr{I}$. If (ψ, ρ) is a non-zero generalized derivation of $\mathscr{A}$ such that $[[\psi(x), x], y] \in \mathscr{T}$, for all $x, y \in \mathscr{I}$, then $\rho(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative.*

DOI: https://dx.doi.org/10.30504/JIMS.2022.335190.1059

Proof. Assume that $[[\psi(x), x], y] \in \mathscr{T}$, for all $x, y \in \mathscr{I}$. Substituting yr for $y, r \in \mathscr{A}$ in the last relation and using this relation, we have $y[[\psi(x), x], r] \in \mathscr{T}$. But $\mathscr{T} \subset \mathscr{I}$. There exists $y_0 \in \mathscr{I}$ such that $y_0 \notin \mathscr{T}$. Thus $y_0\mathscr{A}[[\psi(x), x], r] \subseteq \mathscr{T}$. Since $\mathscr{T}$ is a prime ideal, we get $[[\psi(x), x], r] \in \mathscr{T}$. We find that $\overline{[\psi(x), x]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T})$. By Theorem 2.7, we get $\rho(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative. □

Corollary 2.10. *([1], Theorem 2.2) Let $\mathscr{A}$ be a ring, $\mathscr{I}$ a non-zero ideal of $\mathscr{A}$ and $\mathscr{T}$ a prime ideal of $\mathscr{A}$ such that $\mathscr{T} \subset \mathscr{I}$. If ρ is a non-zero derivation such that $[[\rho(x), x], y] \in \mathscr{T}$, for all $x, y \in \mathscr{I}$, then $\rho(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative.*

Theorem 2.11. *Let $\mathscr{A}$ be a ring, $\mathscr{I}$ a non-zero ideal, $\mathscr{T}$ a prime ideal of $\mathscr{A}$ such that $\mathscr{T} \subset \mathscr{I}$. If (ψ, ρ) is a non-zero generalized derivation of $\mathscr{A}$ such that $\overline{[[\psi(x), x], y]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T})$, for all $x, y \in \mathscr{I}$, then $\rho(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative.*

Proof. Assume that

$$\overline{[[\psi(x), x], y]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T}), \tag{2.16}$$

for all $x, y \in \mathscr{I}$. By linearizing (2.16) and using (2.16), we have

$$\overline{[[\psi(x), t] + [\psi(t), x], y]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T}), \tag{2.17}$$

for all $x, y, t \in \mathscr{I}$. Fix t and putting $\lambda(x) = [\psi(x), t] + [\psi(t), x]$, we get

$$\overline{[\lambda(x), y]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T}), \tag{2.18}$$

for all $x, y \in \mathscr{I}$. Also, we obtain

$$\overline{[\lambda(x), x]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T}), \tag{2.19}$$

for all $x \in \mathscr{I}$. By applying (2.18), we see that

$$[[\lambda(x), y], r] \in \mathscr{T}, \tag{2.20}$$

for all $x \in \mathscr{I}$ and $r \in \mathscr{A}$. Writing xy instead of y in (2.20) and using (2.18) and (2.19), we find that

$$[\lambda(x), x][y, r] + [x, r][\lambda(x), y] \in \mathscr{T}, \tag{2.21}$$

for all $x \in \mathscr{I}$ and $r \in \mathscr{A}$. Replacing r by y in (2.21), we conclude that $[x, y][\lambda(x), y] \in \mathscr{T}$. By applying (2.18) in the last relation, we have $[x, y]\mathscr{A}[\lambda(x), y] \subseteq \mathscr{T}$ and so $[x, y] \in \mathscr{T}$ or $[\lambda(x), y] \in \mathscr{T}$. If $[x, y] \in \mathscr{T}\ \forall\ x, y \in \mathscr{I}$, then $\mathscr{A}/\mathscr{T}$ is commutative by Lemma 2.3. In case $[\lambda(x), y] \in \mathscr{T}$. Putting y by yr, $r \in \mathscr{A}$ in the last relation and using the last equation, we get $y[\lambda(x), r] \in \mathscr{T}$. But $\mathscr{T} \subseteq \mathscr{I}$ and there exists $y_0 \in \mathscr{I}$ such that $y_0 \notin \mathscr{T}$. Thus $y_0[\lambda(x), r] \subseteq \mathscr{T}$. Since $\mathscr{T}$ is a prime ring, we have $[\lambda(x), r] \in \mathscr{T}$. That is

$$\overline{\lambda(x)} \in \mathscr{Z}(\mathscr{A}/\mathscr{T}), \tag{2.22}$$

for all $x \in \mathscr{I}$. Hence

$$\overline{[\psi(x), t] + [\psi(t), x]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T}), \tag{2.23}$$

DOI: https://dx.doi.org/10.30504/JIMS.2022.335190.1059

for all $x, t \in \mathscr{I}$. Suppose that $char(\mathscr{A}/\mathscr{T}) \neq 2$. Taking $t = x$ in (2.23), we have $\overline{[\psi(x), x]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T})$. By Theorem 2.7, we get $\mathscr{A}/\mathscr{T}$ is commutative or $\rho(\mathscr{A}) \subseteq \mathscr{T}$.

Now, if $char(\mathscr{A}/\mathscr{T}) = 2$, the operation $(-)$ will be written $(+)$. Suppose that $\mathscr{I}/\mathscr{T} \cap \mathscr{Z}(\mathscr{A}/\mathscr{T}) = (\overline{0})$. We have $\overline{[[\psi(x), x], y]} = (\overline{0})$ and so $[[\psi(x), x], y] \in \mathscr{T}$. By Theorem 2.9, we get $\rho(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative. We will assume that $\mathscr{I}/\mathscr{T} \cap \mathscr{Z}(\mathscr{A}/\mathscr{T}) \neq (\overline{0})$. There exists $(\overline{0}) \neq \overline{z} \in \mathscr{I}/\mathscr{T} \cap \mathscr{Z}(\mathscr{A}/\mathscr{T})$. Thus $z \notin \mathscr{T}$. By applying (2.23), we have

$$[[\psi(x), t] + [\psi(t), x], r] \in \mathscr{T}, \tag{2.24}$$

for all $x, t \in \mathscr{I}$ and $r \in \mathscr{A}$. Replacing x by xy in (2.24) and using this equation, we get

$$[\psi(x)[y, t] + x[\rho(y), t] + [x, t]\rho(y) + x[\psi(t), y], r] \in \mathscr{T}, \tag{2.25}$$

for all $x, y, t \in \mathscr{I}$ and $r \in \mathscr{A}$. Since $\mathscr{I}/\mathscr{T} \cap \mathscr{Z}(\mathscr{A}/\mathscr{T}) \neq (\overline{0})$, we can putting $t = z$ in (2.25), where $(\overline{0}) \neq \overline{z} \in \mathscr{I}/\mathscr{T} \cap \mathscr{Z}(\mathscr{A}/\mathscr{T})$. We obtain that $[x[\psi(z), y], r] \in \mathscr{T}$ but $\overline{[\psi(z), y]} = \overline{\lambda(z)} \in \mathscr{Z}(\mathscr{A}/\mathscr{T})$ from (2.22). That is $[\psi(z), y][x, r] \in \mathscr{T}$. We see that $[\psi(z), y]\mathscr{A}[x, r] \subseteq \mathscr{T}$. We obtain that $[\psi(z), y] \in \mathscr{T}$ or $[x, r] \in \mathscr{T}$. If $[x, r] \in \mathscr{T}$, then $\mathscr{A}/\mathscr{T}$ is commutative by Lemma 2.3. If $[\psi(z), y] \in \mathscr{T}$, then $\overline{\psi(z)} \in \mathscr{Z}(\mathscr{I}/\mathscr{T})$. By Lemma 2.4, we get

$$\overline{\psi(z)} \in \mathscr{Z}(\mathscr{A}/\mathscr{T}). \tag{2.26}$$

Replacing x by z in (2.25), where $(\overline{0}) \neq \overline{z} \in \mathscr{I}/\mathscr{T} \cap \mathscr{Z}(\mathscr{A}/\mathscr{T})$, we see that

$$[\psi(z)[y, t] + z[\rho(y), t] + z[\psi(t), y], r] \in \mathscr{T}, \tag{2.27}$$

for all $y, t \in \mathscr{I}$ and $r \in \mathscr{A}$. Substituting zt for t in (2.27) and using (2.26), we find that

$$z^2[[\rho(y), t] + [\rho(t), y], r] \in \mathscr{T},$$

and so

$$[[\rho(y), t] + [\rho(t), y], r] \in \mathscr{T}, \tag{2.28}$$

for all $y, t \in \mathscr{I}$ and $r \in \mathscr{A}$. By applying (2.28), we have

$$\begin{aligned}&[xy + yx, \rho(x)] + [x^2, \rho(y)]\\&\quad = x([y, \rho(x)] + [x, \rho(y)]) + ([y, \rho(x)] + [x, \rho(y)])x + y[x, \rho(x)] + [x, \rho(x)]y \in \mathscr{T},\end{aligned}$$

for all $x, y \in \mathscr{I}$. Hence

$$[xy + yx, \rho(x)] + [x^2, \rho(y)] \in \mathscr{T}, \tag{2.29}$$

for all $x, y \in \mathscr{I}$. Putting $y = x$ in (2.28) and using $char(\mathscr{A}/\mathscr{T}) = 2$, we get

$$[x^2, \rho(x)] \in \mathscr{T}, \tag{2.30}$$

for all $x, y \in \mathscr{I}$. Writing yx instead of y in (2.29) and using (2.29) and (2.30), we obtain

$$(xy + yx)[x, \rho(x)] + [x^2, y]\rho(x) \in \mathscr{T}, \tag{2.31}$$

DOI: https://dx.doi.org/10.30504/JIMS.2022.335190.1059

for all $x, y \in \mathscr{I}$. Replacing y by ry in (2.31), where $r \in \mathscr{A}$, we see that

$$(xry + ryx)[x, \rho(x)] + r[x^2, y]\rho(x) + [x^2, r]y\rho(x) \in \mathscr{T}, \tag{2.32}$$

for all $x, y \in \mathscr{I}$ and $r \in \mathscr{A}$. Multiplying the equation (2.31) on the left for r, we find that

$$(rxy + ryx)[x, \rho(x)] + r[x^2, y]\rho(x) \in \mathscr{T}, \tag{2.33}$$

for all $x, y \in \mathscr{I}$ and $r \in \mathscr{A}$. Using (2.32) and (2.33), we conclude that

$$[x, r]y[x, \rho(x)] + [x^2, r]y\rho(x) \in \mathscr{T}.$$

Taking $r = \rho(x)$ in last relation and using (2.30), we have $[x, \rho(x)]y[x, \rho(x)] \in \mathscr{T}$ and so $[x, \rho(x)] \in \mathscr{T}$. By Corollary 2.6, we get $\rho(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative. $\square$

Corollary 2.12. *Let $\mathscr{A}$ be a ring, $\mathscr{I}$ a non-zero ideal, $\mathscr{T}$ a prime ideal of $\mathscr{A}$ such that $\mathscr{T} \subset \mathscr{I}$. If ρ is a non-zero derivation such that $\overline{[[\rho(x), x], y]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T})$, for all $x, y \in \mathscr{I}$, then $\rho(\mathscr{A}) \subseteq \mathscr{T}$ or $\mathscr{A}/\mathscr{T}$ is commutative.*

Example 2.13. *Consider the ring $\mathscr{A} = \mathbb{Q}[X] \times \mathscr{S}$, where $\mathscr{S}$ any non-commutative ring. Let $\psi = \rho : \mathscr{A} \to \mathscr{A}$ defined by $\psi(q(X), s) = (q(X)', 0)$ be a (generalized) derivation. Let $\mathscr{T} = (0, \mathscr{J})$ where $\mathscr{J}$ any proper ideal of $\mathscr{S}$ and $\mathscr{I} = \mathscr{A}$. We see that (i) $[\psi(x), x] \in \mathscr{T}$, (ii) $[[\psi(x), x], y] \in \mathscr{T}$, (iii) $\overline{[\psi(x), x]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T})$, (iv) $\overline{[[\psi(x), x], y]} \in \mathscr{Z}(\mathscr{A}/\mathscr{T})$ $\forall\ x, y \in \mathscr{I}$. But $\mathscr{T}$ is not prime ideal of $\mathscr{A}$, also $\rho(\mathscr{A}) \nsubseteq \mathscr{T}$ and $\mathscr{A}/\mathscr{T}$ is non-commutative.*

Acknowledgments

The authors wishes to thank the referee for his/her valuable comments and suggestion.

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DOI: https://dx.doi.org/10.30504/JIMS.2022.335190.1059