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HANKEL OPERATORS ON BERGMAN SPACES INDUCED BY REGULAR WEIGHTS

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Abstract. In this paper, given two regular weights ω, Ω, we characterize these symbols $f \in L^1_\Omega$ for which the induced Hankel operators H^Ω_f are bounded (or compact) from weighted Bergman space A^p_ω to Lebesgue space L^q_Ω for all $1 < p, q < \infty$. Moreover, we answer a question posed by X. Lv and K. Zhu [Integr. Equ. Oper. Theory, 91(2019), 91:5] in the case $n = 1$.

1. Introduction

Let $\mathbb{D}$ be the unit disc in the complex plane. Given some non-negative integrable functions ω on $\mathbb{D}$ and $1 \le p < \infty$, the space L^p_ω (or $L^p(\omega dA)$) consists of all Lebesgue measurable functions f in $\mathbb{D}$ such that

$$\|f\|_{L^p_\omega} = \left(\int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z)\right)^{\frac{1}{p}} < \infty,$$

where dA is the normalized area measure on $\mathbb{D}$. We use L^p to stand the usual p-th Lebesgue space with the norm $\|\cdot\|_{L^p} = \left(\int_{\mathbb{D}} |\cdot|^p dA\right)^{\frac{1}{p}}$. Let $H(\mathbb{D})$ denote the space of analytic functions in $\mathbb{D}$. The weighted Bergman space is defined as $A^p_\omega = L^p_\omega \cap H(\mathbb{D})$.

As mentioned in [22], a radial weight ω belongs to the class $\hat{\mathcal{D}}$ if $\omega \in L^1[0,1)$ and $\hat{\omega}(z) = \int_{|z|}^1 \omega(s) ds$ satisfies the doubling condition that $\hat{\omega}(r) \le K \hat{\omega}(\frac{1+r}{2})$ for all $0 \le r < 1$, where K is a constant

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independent of r. Furthermore, we say ω belongs to the class $\mathcal{R}$ of regular radial weights if $\omega \in \hat{\mathcal{D}}$ and satisfies

$$\omega(r) \simeq \frac{\int_r^1 \omega(s)ds}{1-r} \text{ for } 0 \le r < 1.$$

Given ω radial we can extend it to $\mathbb{D}$ with $\omega(z) = \omega(|z|)$. Notice the classical weight $\omega(r) = (1-r^2)^\alpha$ with $\alpha > -1$ and the weights in [1] and [9] are regular. Recently, Bergman spaces A^p_ω with $\omega \in \mathcal{R}$ are studied in [10, 18, 20–22].

The space A^p_ω is closed in L^p_ω. A^2_ω is a Hilbert space with the inner product

$$\langle f, g\rangle_\omega = \int_{\mathbb{D}} f(z)\overline{g(z)}\omega(z)dA(z), \quad f, g \in A^2_\omega.$$

for any fixed $z \in \mathbb{D}$, the mapping $f \mapsto f(z)$ is a bounded linear functional on A^2_ω. By the Riesz representation theorem in functional analysis, there exists a unique function $B_z \in A^2_\omega$ such that $f(z) = \langle f, B_z\rangle_\omega$ for all $f \in A^2_\omega$. The function B_z is called the Bergman kernel of A^2_ω. The orthogonal projection P_ω from L^2_ω to A^2_ω can be represented as

$$P_\omega(g)(z) = \int_{\mathbb{D}} g(\zeta)\overline{B_z(\zeta)}\omega(\zeta)dA(\zeta).$$

With this expression, P_ω can be extended to a bounded linear operator from L^p_ω to A^p_ω for all $1 \le p < \infty$. The Hankel operator H^ω_f with symbol f is defined by

$$H^\omega_f(g) = (Id - P_\omega)(fg),$$

where Id is the identity operator.

The study of Hankel operators on Bergman spaces initialed from [4] where Axler characterized the boundedness and compactness of Hankel operators induced by conjugate analytic functions. Later on, Axler's result was generalized in [2, 3] to weighted Bergman spaces of the unit ball in $\mathbb{C}^n$. For general symbol functions, Zhu in [23] first established the connection between size estimates of Hankel operators and the mean oscillation of the symbols in the Bergman metric. This idea was further investigated in the context of bounded symmetric domains and strongly pseudo convex domains, see [5–7, 12, 13]. In [16], based on the Hilbert space property, Pau characterized those $f \in L^2_{(1-|\cdot|^2)^\beta}$ such that H_f is bounded on $A^2_{(1-|\cdot|^2)^\beta}$. On the unit ball of $\mathbb{C}^n$, restricted themselves on $1 < p \le q < \infty$, Pau, Zhao and Zhu in [17] obtained the characterization on $f \in L^q_{(1-|\cdot|^2)^\beta}$ such that H_f and $H_{\overline{f}}$ are both bounded (or compact) from $A^p_{(1-|\cdot|^2)^\alpha}$ to $L^q_{(1-|\cdot|^2)^\beta}$. Then Lv and Zhu in [14] discussed the same question in the case $1 < q < p < \infty$ under the restriction "$pn < q(n+1+\alpha)$", Later on, for $1 < p \le q < \infty$, Peláez, Perälä and Rättyä in [19] obtained the the condition on $f \in L^q_\nu$ such that both $H_f, H_{\overline{f}} : A^p_\omega \to L^q_\nu$ are bounded, where $\nu \in \mathcal{B}_q$ being radial and $\omega \in \mathcal{D}$. More recently, for regular weight ω, Hu and Lu in [10] characterized those $f \in L^1_\omega$ such that, for given $1 < p, q < \infty$, H_f is bounded (or compact) from A^p_ω to L^q_ω.

The purpose of this paper is to extend Hu and Lu's results to the case of different regular weights. To be precise, given $\omega, \Omega \in \mathcal{R}$, we characterize these symbols $f \in L^1_\Omega$ for which the induced Hankel

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operators H_f^Ω are bounded (or compact) from A^p_ω to L^q_Ω for all possible $1<p,q<\infty$. We summarize the main results of the paper as below:

Theorem 1.1. *Let $\omega,\Omega\in\mathcal{R}$, $1<p\le q<\infty$. Then for $f\in L^1_\Omega$, the following statements are equivalent:*

(A) *$H_f^\Omega : A^p_\omega\to L^q_\Omega$ is bounded;*

(B) *For some (or any) $0<r\le\alpha$, $\breve{\Omega}^{\frac1q}\breve{\omega}^{-\frac1p}G_{q,r}(f)\in L^\infty$;*

(C) *f admits a decomposition $f=f_1+f_2$, where $f_1\in C^1(\mathbb{D})$ satisfies*

$$(1-|\cdot|)\breve{\Omega}^{\frac1q}\breve{\omega}^{-\frac1p}|\overline{\partial}f_1|\in L^\infty, \tag{1.1}$$

and f_2 has the property that, for some (or any) $r>0$,

$$\breve{\Omega}^{\frac1q}\breve{\omega}^{-\frac1p}M_r(|f_2|^q)^{\frac1q}\in L^\infty. \tag{1.2}$$

Moreover, for $0<r\le\alpha$,

$$\|H_f^\Omega\|_{A^p_\omega\to L^q_\Omega}\simeq\left\|\breve{\Omega}^{\frac1q}\breve{\omega}^{-\frac1p}G_{q,r}(f)\right\|_{L^\infty} \tag{1.3}$$

Theorem 1.2. *Let $\omega,\Omega\in\mathcal{R}$, $1<p\le q<\infty$. Then for $f\in L^1_\Omega$, the following statements are equivalent:*

(A) *$H_f^\Omega : A^p_\omega\to L^q_\Omega$ is compact;*

(B) *For some (or any) $0<r\le\alpha$, $\lim_{|z|\to1}\breve{\Omega}(z)^{\frac1q}\breve{\omega}(z)^{-\frac1p}G_{q,r}(f)(z)=0$;*

(C) *f admits a decomposition $f=f_1+f_2$, where $f_1\in C^1(\mathbb{D})$ satisfies*

$$\lim_{|z|\to1}(1-|z|)\breve{\Omega}(z)^{\frac1q}\breve{\omega}(z)^{-\frac1p}|\overline{\partial}f_1(z)|=0,$$

and f_2 satisfies

$$\lim_{|z|\to1}\breve{\Omega}(z)^{\frac1q}\breve{\omega}(z)^{-\frac1p}M_r(|f_2|^q)(z)^{\frac1q}=0$$

for some (or any) $r>0$.

Theorem 1.3. *Let $\omega,\Omega\in\mathcal{R}$, $1<q<p<\infty$. Then for $f\in L^1_\Omega$, the following statements are equivalent:*

(A) *$H_f^\Omega : A^p_\omega\to L^q_\Omega$ is bounded;*

(B) *$H_f^\Omega : A^p_\omega\to L^q_\Omega$ is compact;*

(C) *For some (or any) $0<r\le\alpha/2$, $\Omega^{\frac1q}\omega^{-\frac1p}G_{q,r}(f)\in L^{\frac{pq}{p-q}}$;*

(D) *f admits a decomposition $f=f_1+f_2$, where*

$$f_1\in C^1(\mathbb{D}),\ \Omega^{\frac1q}\omega^{-\frac1p}(1-|\cdot|)|\overline{\partial}f_1|\in L^{\frac{pq}{p-q}}, \tag{1.4}$$

and

$$\Omega^{\frac1q}\omega^{-\frac1p}M_r(|f_2|^q)^{\frac1q}\in L^{\frac{pq}{p-q}} \tag{1.5}$$

for some (or any) $r>0$. Moreover,

$$\|H_f^\Omega\|_{A^p_\omega\to L^q_\Omega}\simeq\left\|\Omega^{\frac1q}\omega^{-\frac1p}G_{q,r}(f)\right\|_{L^{\frac{pq}{p-q}}}. \tag{1.6}$$

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The paper is organized as follows. In Section 2, we will give some preliminaries. Section 3 is devoted to give the proofs of our main theorems. Throughout this paper, we use C to denote positive constants whose value may change from line to line, but do not depend on functions being considered. For two quantities A and B, we write $A \lesssim B$ if there exists some C such that $A \leq CB$. We call A and B are equivalent, denoted by $A \simeq B$, if $A \lesssim B \lesssim A$.

2. Preliminaries

We begin by stating some known results which are used in the proof of the main results. Let

$$\beta(z,\xi) = \frac{1}{2}\log\frac{1+|\varphi_z(\xi)|}{1-|\varphi_z(\xi)|}$$

be the Bergman distance of z, ξ in $\mathbb{D}$, where $\varphi_z(\xi) = \frac{\xi - z}{1-\bar{z}\xi}$. For $z \in \mathbb{D}$ and $r > 0$, let $D(z,r) = \{w \in \mathbb{D} : \beta(z,w) < r\}$ denote the Bergman disk with center z and radius r. Given $\omega \in \mathcal{R}$, write

$$\breve{\omega}(z) = (1-|z|)^2\omega(z).$$

The following lemma exhibits some basic estimates. See [22] for more details.

Lemma 2.1. *Suppose $\omega \in \mathcal{R}$, then*

(A) *For $1 < p < \infty$, there holds*

$$\|B_z\|_{A^p_\omega} \simeq \breve{\omega}(z)^{\frac{1}{p}-1}.$$

(B) *There is some positive constant α such that*

$$|B_z(\xi)| \simeq B_z(z) \simeq B_\xi(\xi) \simeq \frac{1}{\breve{\omega}(z)} \simeq \frac{1}{\breve{\omega}(\xi)},$$

for all $z, \xi \in \mathbb{D}$ with $\beta(z,\xi) < r$.

(C) *For $r > 0$ fixed,*

$$\omega(D(z,r)) := \int_{D(z,r)} \omega dA \simeq \breve{\omega}(z).$$

For $1 < p < \infty$, write $b_{p,z} = B_z/\|B_z\|_{A^p_\omega}$. Clearly, $\|b_{p,z}\|_{A^p_\omega} = 1$. By Lemma 2.1 we know

$$\inf_{\xi\in D(z,\alpha)} |b_{p,z}(\xi)| \simeq \sup_{\xi\in D(z,\alpha)} |b_{p,z}(\xi)| \simeq \breve{\omega}(z)^{-\frac{1}{p}}. \tag{2.1}$$

For any $r > 0$, [8, Lemma 2.13] tells us there exists a sequence $\{z_j\}_{j=1}^\infty$ such that

$$\mathbb{D} = \cup_{j=1}^\infty D(z_j, r), \quad D\left(z_j, \frac{r}{4}\right) \cap D\left(z_k, \frac{r}{4}\right) = \emptyset \ \text{ for } \ j \neq k.$$

Such a sequence is called an r-lattice of $\mathbb{D}$. For $E \subseteq \mathbb{D}$ Lebesgue measurable, let χ_E be the characteristic function of E, and write $|E| = \int_{\mathbb{D}} \chi_E dA$. Given some r-lattice $\{z_j\}_{j=1}^\infty$ and $R > 0$, we have some constant N such that

$$\sum_{j=1}^\infty \chi_{D(z_j,R)} \leq N. \tag{2.2}$$

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Given $r>0$, the local mean operator M_r on L^1_{loc} is defined as

$$M_r(f)(z)=\frac{1}{|D(z,r)|}\int_{D(z,r)}f(\xi)dA(\xi).$$

For any $r>0$, M_r is a bounded linear operator on L^p_ω for $1\le p\le\infty$.

Let $q\ge 1$ and $r>0$. For $f\in L^q_{loc}$, define $G_{q,r}(f)$ to be

$$G_{q,r}(f)(z)=\inf\left\{\left(\frac{1}{|D(z,r)|}\int_{D(z,r)}|f-h|^q dA\right)^{\frac{1}{q}}:h\in H(D(z,r))\right\},\quad z\in\mathbb{D}.$$

Since $\omega\in\mathcal{R}$,

$$G_{q,r}(f)(z)\simeq\inf\left\{\left(\frac{1}{\omega(D(z,r))}\int_{D(z,r)}|f-h|^q\omega dA\right)^{\frac{1}{q}}:h\in H(D(z,r))\right\}.$$

Let μ be a finite positive Borel measure on $\mathbb{D}$, μ is called a q-Carleson measure for A^p_ω if the identity operator $Id:A^p_\omega\to L^q(\mu)$ is bounded, μ is called a vanishing q-Carleson measure for A^p_ω if the identity operator $Id:A^p_\omega\to L^q(\mu)$ is compact.

The following two lemmas give the equivalent conditions of a positive Borel measure μ belonging a (vanishing) q-Carleson measure for A^p_ω. See [20] and [22] respectively.

Lemma 2.2. *Suppose $1<p\le q<\infty$, $\omega\in\mathcal{R}$, and let μ be a positive Borel measure on $\mathbb{D}$. Then the following assertions hold:*

(A) *μ is a q-Carleson measure for A^p_ω if and only if*

$$\sup_{z\in\mathbb{D}}\frac{\mu(D(z,r))}{\omega(D(z,r))^{\frac{q}{p}}}<\infty \tag{2.3}$$

for some (or any) $r\in(0,\alpha]$. Moreover,

$$\|Id\|^q_{A^p_\omega\to L^q(d\mu)}\simeq\sup_{z\in\mathbb{D}}\frac{\mu(D(z,r))}{\omega(D(z,r))^{\frac{q}{p}}}.$$

(B) *μ is a vanishing q-Carleson measure for A^p_ω if and only if*

$$\lim_{|z|\to 1}\frac{\mu(D(z,r))}{\omega(D(z,r))^{\frac{q}{p}}}=0$$

for some (or any) $r\in(0,\alpha]$.

Lemma 2.3. *Suppose $1<q<p<\infty$, $\omega\in\mathcal{R}$, and let μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(A) *μ is a q-Carleson measure for A^p_ω;*

(B) *μ is a vanishing q-Carleson measure for A^p_ω;*

(C) *For some (or any) $r\in(0,\alpha]$, $\mu(D(\cdot,r))/\omega(D(\cdot,r))\in L^{\frac{p}{p-q}}_\omega$.*

Furthermore,

$$\|Id\|_{A^p_\omega\to L^q(d\mu)}\simeq\|\mu(D(\cdot,r))/\omega(D(\cdot,r))\|_{L^{\frac{p}{p-q}}_\omega}.$$

The following lemmas was proved in [10], which are important in the proof of the main results.

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Lemma 2.4. *Suppose* $1<p<\infty$ *and* $\omega\in\mathcal{R}$. *For* $f\in L^p((1-|\cdot|)^p\omega dA)$, *let*

$$u(z)=\sum_{j=1}^{\infty}B_{z_j}(z)\int_{\mathbb{D}}\frac{\psi_j(\xi)}{(\xi-z)B_{z_j}(\xi)}f(\xi)dA(\xi).$$

Then u *weakly solves the equation* $\overline{\partial}u=f$ *in* $\mathbb{D}$, *and there is some constant* C, *independent of* f, *such that*

$$\|u\|_{L^p_\omega}\le C\|f\|_{L^p_{(1-|\cdot|)^p\omega}}.$$

Lemma 2.5. *Suppose* $\Omega\in\mathcal{R}$, $f\in L^1_\Omega$ *and* $\overline{\partial}f\in L^p_{(1-|\cdot|)^p\Omega}$ *for some* $p>1$. *Then for* $g\in H^\infty$ *there holds*

$$H_f^\Omega(g)=u-P_\Omega(u),$$

where

$$u(z)=\sum_{j=1}^{\infty}B_{z_j}(z)\int_{\mathbb{D}}\frac{\psi_j(\xi)}{(\xi-z)B_{z_j}(\xi)}g(\xi)\overline{\partial}f(\xi)dA(\xi).$$

3. Proof of the main results

Now we are ready to prove Theorems 1.1-1.3.

Proof of Theorem 1.1. For $1<p<\infty$, H^∞ is dense in A^p_ω. So for $f\in L^1_\Omega$, H_f^Ω is well (densely) defined on A^p_ω.

$(A)\Rightarrow(B)$. Let $r\in(0,\alpha]$ be fixed, (2.1) tells us

$$\inf_{\xi\in D(z,\alpha)}|b_{p,z}(\xi)|\gtrsim b_{p,z}(z)\simeq\breve{\omega}(z)^{-\frac{1}{p}}>0$$

for $z\in\mathbb{D}$. Thus, $\frac{1}{b_{p,z}}P_\Omega(fb_{p,z})\in H(D(z,r))$. This together with Lemma 2.1 shows

$$\begin{aligned}\left\|H_f^\Omega(b_{p,z})\right\|_{L^q_\Omega}^q&=\int_{\mathbb{D}}|f(\xi)b_{p,z}(\xi)-P_\Omega(fb_{p,z})(\xi)|^q\,\Omega(\xi)dA(\xi)\\&\ge\int_{D(z,r)}|b_{p,z}(\xi)|^q\left|f(\xi)-\frac{1}{b_{p,z}(\xi)}P_\Omega(fb_{p,z})(\xi)\right|^q\Omega(\xi)dA(\xi)\\&\gtrsim b_{p,z}(z)^q\int_{D(z,r)}\left|f(\xi)-\frac{1}{b_{p,z}(\xi)}P_\Omega(fb_{p,z})(\xi)\right|^q\Omega(\xi)dA(\xi)\\&\gtrsim b_{p,z}(z)^q\Omega(D(z,r))G_{q,r}(f)(z)^q\\&\simeq\breve{\omega}(z)^{-\frac{q}{p}}\breve{\Omega}(z)G_{q,r}(f)(z)^q.\end{aligned}\tag{3.1}$$

On the other hand, the boundedness of $H_f^\Omega:A^p_\omega\to L^q_\Omega$ shows

$$\left\|H_f^\Omega(b_{p,z})\right\|_{L^q_\Omega}^q\le\left\|H_f^\Omega\right\|_{A^p_\omega\to L^q_\Omega}^q\|b_{p,z}\|_{A^p_\omega}^q=\left\|H_f^\Omega\right\|_{A^p_\omega\to L^q_\Omega}^q.$$

Therefore, we obtain

$$\breve{\omega}(z)^{-\frac{1}{p}}\breve{\Omega}(z)^{\frac{1}{q}}G_{q,r}(f)(z)\lesssim\|H_f^\Omega\|_{A^p_\omega\to L^q_\Omega}.\tag{3.2}$$

From this we know, $\breve{\Omega}^{\frac{1}{q}}\breve{\omega}^{-\frac{1}{p}}G_{q,r}(f)\in L^\infty(\mathbb{D})$.

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$(B) \Rightarrow (C)$. Suppose $\left\|\breve{\Omega}^{\frac{1}{q}}\breve{\omega}^{-\frac{1}{p}}G_{q,r}(f)\right\|_{L^\infty} < \infty$ for some $r \in (0, \alpha]$. Fix $\{z_j\}_{j=1}^{\infty}$ to be an $\frac{r}{2}$-lattice, and take $\{\psi_j\}$ to be a partition of unity subordinate to $\{D(z_j, r/2)\}$, satisfying $|(1-|z_j|)\overline{\partial}\psi_j| \leq C$. A normal family argument shows that for each j, there exists some $h_j \in H(D(z_j, r))$ such that

$$\frac{1}{\Omega(D(z_j,r))}\int_{D(z_j,r)} |f-h_j|^q \Omega dA = G_{q,r}(f)(z_j)^q.$$

Set

$$f_1(z) = \sum_{j=1}^{\infty} h_j(z)\psi_j(z) \in C^\infty(\mathbb{D})$$

and $f_2 = f - f_1$. For $z \in \mathbb{D}$, set $J_z = \{j : z \in D(z_j, r)\}$. Then, $1-|z_j| \simeq 1-|z|$ for $j \in J_z$. And

$$|J_z| := \sum_{j=1}^{\infty} \chi_{D(z_j,r)}(z) \leq C.$$

A similar way to that on pages 254-255 in [15], for $z \in \mathbb{D}$, there holds

$$\left|\overline{\partial} f_1(z)\right| \lesssim \frac{1}{1-|z|}\sum_{j\in J_z} G_{q,r}(f)(z_j). \tag{3.3}$$

This shows

$$(1-|z|)\breve{\Omega}(z)^{\frac{1}{q}}\breve{\omega}(z)^{-\frac{1}{p}}|\overline{\partial} f_1(z)| \lesssim \left\|\breve{\Omega}^{\frac{1}{q}}\breve{\omega}^{-\frac{1}{p}}G_{q,r}(f)\right\|_{L^\infty}. \tag{3.4}$$

Thus (1.1) follows.

For f_2, we have

$$\begin{aligned}\left(\frac{1}{|D(z,r)|}\int_{D(z,r)} |f_2|^q\, dA\right)^{\frac{1}{q}} &= \sum_{j=1}^{\infty}\left(\frac{1}{|D(z,r)|}\int_{D(z,r)} |(f-h_j)\psi_j|^q\, dA\right)^{\frac{1}{q}} \\ &= \sum_{j=1}^{\infty}\left(\frac{1}{|D(z,r)|}\int_{D(z,r)\cap D(z_j,r/2)} |f-h_j|^q\, dA\right)^{\frac{1}{q}}.\end{aligned} \tag{3.5}$$

Hence,

$$\breve{\Omega}(z)^{\frac{1}{q}}\breve{\omega}(z)^{-\frac{1}{p}}M_r(|f_2|^q)(z)^{\frac{1}{q}} \lesssim \left\|\breve{\Omega}^{\frac{1}{q}}\breve{\omega}^{-\frac{1}{p}}G_{q,r}(f)\right\|_{L^\infty}. \tag{3.6}$$

This shows (1.2) holds. Notice that (1.2) is independent of precise values of r, and different values of r give equivalent norms $\left\|\breve{\Omega}^{\frac{1}{q}}\breve{\omega}^{-\frac{1}{p}}M_r(|f_2|^q)^{\frac{1}{q}}\right\|_{L^\infty}$. We finish the proof of implication $(B) \Rightarrow (C)$.

$(C) \Rightarrow (A)$. Set $d\mu = |f_2|^q \Omega dA$. We show $d\mu$ is a q-Carleson measure for A^p_ω. Since $\Omega(\xi) \simeq \Omega(z)$ for $\xi \in D(z,r)$, we have

$$\begin{aligned}\int_{D(z,r)} |f_2(\xi)|^q\, \Omega(\xi) dA(\xi) &\simeq \Omega(z)\int_{D(z,r)} |f_2(\xi)|^q\, dA(\xi) \\ &\simeq \frac{\Omega(D(z,r))}{|D(z,r)|}\int_{D(z,r)} |f_2(\xi)|^q\, dA(\xi).\end{aligned}$$

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Thus,

$$\begin{aligned}\frac{\mu(D(z,r))}{\omega(D(z,r))^{\frac{q}{p}}} &= \frac{\int_{D(z,r)} |f_2(\xi)|^q\,\Omega(\xi)dA(\xi)}{\omega(D(z,r))^{\frac{q}{p}}}\\ &\simeq \frac{\Omega(D(z,r))}{\omega(D(z,r))^{\frac{q}{p}}}\cdot\frac{1}{|D(z,r)|}\int_{D(z,r)}|f_2(\xi)|^q\,dA(\xi)\\ &\simeq \breve{\Omega}(z)\breve{\omega}(z)^{-\frac{q}{p}}M_r(|f_2|^q)(z).\end{aligned}$$

It follows from (1.2) that

$$\sup_{z\in\mathbb{D}}\frac{\mu(D(z,r))}{\omega(D(z,r))^{\frac{q}{p}}}<\infty.$$

Thus Lemma 2.2 shows $d\mu$ is a q-Carleson measure for A^p_ω. Moreover,

$$\|Id\|_{A^p_\omega\to L^q(d\mu)}\simeq\sup_{z\in\mathbb{D}}\frac{\mu(D(z,r))^{\frac{1}{q}}}{\omega(D(z,r))^{\frac{1}{p}}}\simeq\sup_{z\in\mathbb{D}}\breve{\Omega}(z)^{\frac{1}{q}}\breve{\omega}(z)^{-\frac{1}{p}}M_r(|f_2|^q)^{\frac{1}{q}}(z).$$

Therefore,

$$\begin{aligned}\left\|H^\Omega_{f_2}g\right\|_{L^q_\Omega}&\lesssim\|f_2g\|_{L^q_\Omega}=\left(\int_{\mathbb{D}}|f_2|^q|g|^q\Omega dA\right)^{\frac{1}{q}}\\ &\lesssim\|Id\|_{A^p_\omega\to L^q(d\mu)}\|g\|_{L^p_\omega}\simeq\left\|\breve{\Omega}^{\frac{1}{q}}\breve{\omega}^{-\frac{1}{p}}M_r(|f_2|^q)^{\frac{1}{q}}\right\|_{L^\infty}\|g\|_{L^p_\omega}.\end{aligned}\tag{3.7}$$

Now we suppose f_1 satisfies (1.1). For $g\in H^\infty$, pick u as in Lemma 2.5 that

$$u(z)=\sum_{j=1}^{\infty}B_{z_j}(z)\int_{\mathbb{D}}\frac{\psi_j(\xi)}{(\xi-z)B_{z_j}(\xi)}g(\xi)\bar\partial f_1(\xi)dA(\xi).$$

Then Lemma 2.4 and Lemma 2.5 give

$$H^\Omega_{f_1}(g)=u-P_\Omega(u)\ \text{ and }\ \|u\|_{L^q_\Omega}\lesssim\|(1-|\cdot|)g\bar\partial f_1\|_{L^q_\Omega}.$$

By the boundedness of P_Ω on L^q_Ω we obtain

$$\left\|H^\Omega_{f_1}g\right\|_{L^q_\Omega}\le(1+\|P_\Omega\|_{L^q_\Omega})\|u\|_{L^q_\Omega}\lesssim\left\|(1-|\cdot|)g\bar\partial f_1\right\|_{L^q_\Omega}.$$

Meanwhile, we know from (1.1) and Lemma 2.2 that $d\nu=\left[(1-|\cdot|)|\bar\partial f_1|\right]^q\Omega dA$ is a q-Carleson measure for A^p_ω, and the Carleson constant of $d\nu$ is less than or equal to $C\left\|(1-|\cdot|)\breve{\Omega}^{\frac{1}{q}}\breve{\omega}^{-\frac{1}{p}}\left|\bar\partial f_1\right|\right\|_{L^\infty}$. Then

$$\left\|(1-|\cdot|)g\bar\partial f_1\right\|_{L^q_\Omega}\lesssim\left\|(1-|\cdot|)\breve{\Omega}^{\frac{1}{q}}\breve{\omega}^{-\frac{1}{p}}\left|\bar\partial f_1\right|\right\|_{L^\infty}\cdot\|g\|_{L^p_\omega}.$$

Hence, we have

$$\left\|H^\Omega_{f_1}g\right\|_{L^q_\Omega}\lesssim\left\|(1-|\cdot|)\breve{\Omega}^{\frac{1}{q}}\breve{\omega}^{-\frac{1}{p}}\left|\bar\partial f_1\right|\right\|_{L^\infty}\cdot\|g\|_{L^p_\omega}.$$

With this and (3.7) we obtain

$$\left\|H^\Omega_f\right\|_{A^p_\omega\to L^q_\Omega}\lesssim\left\{\left\|(1-|\cdot|)\breve{\Omega}^{\frac{1}{q}}\breve{\omega}^{-\frac{1}{p}}|\bar\partial f_1|\right\|_{L^\infty}+\left\|\breve{\Omega}^{\frac{1}{q}}\breve{\omega}^{-\frac{1}{p}}M_r(|f_2|^q)^{\frac{1}{q}}\right\|_{L^\infty}\right\}.\tag{3.8}$$

This finishes the proof of the equivalence among $(A),(B)$ and (C). Moreover, the desired norm estimate (1.3) come from (3.2), (3.4), (3.6) and (3.8). □

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Proof of Theorem 1.2. Suppose $H_f^\Omega : A^p_\omega \to L^q_\Omega$ is compact. Since $b_{p,z}$ tends to 0 weakly in A^p_ω, for $0 < r \leq \alpha$, from (3.1) we have

$$\breve{\omega}(z)^{-\frac{q}{p}}\breve{\Omega}(z)G_{q,r}(f)(z)^q \lesssim \left\|H_f^\Omega(b_{p,z})\right\|_{L^q_\Omega} \to 0$$

as $|z| \to 1$. So, (A) implies (B).

Suppose (B) is valid. From (3.3) and (3.6) we have

$$\breve{\Omega}(z)^{\frac{1}{q}}\breve{\omega}(z)^{-\frac{1}{p}}(1-|z|)|\overline{\partial}f_1(z)| \lesssim \sum_{j\in J_z}\breve{\Omega}(z_j)^{\frac{1}{q}}\breve{\omega}(z_j)^{-\frac{1}{p}}G_{q,r}(f)(z_j)$$

and

$$\breve{\Omega}(z)^{\frac{1}{q}}\breve{\omega}(z)^{-\frac{1}{p}}M_r(|f_2|^q)(z)^{\frac{1}{q}} \leq C\sum_{j\in J_z}\breve{\Omega}(z_j)^{\frac{1}{q}}\breve{\omega}(z_j)^{-\frac{1}{p}}G_{q,r}(f)(z_j).$$

From these the statement (C) follows.

Now we prove the implication $(C) \Rightarrow (A)$. As that in the proof of Theorem 1.1, we know $d\mu = |f_2|^q\Omega dA$ and $d\nu = \left[(1-|\cdot|)|\overline{\partial}f_1|\right]^q\Omega dA$ are both vanishing q-Carleson measures for A^p_ω. For any bounded sequence $\{g_m\}_{m=1}^\infty$ in A^p_ω with $g_m \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $m \to \infty$, we have

$$\left\|H_{f_1}^\Omega(g_m)\right\|_{L^q_\Omega} \lesssim \left\|\left[(1-|\cdot|)|\overline{\partial}f_1|\right]g_m\right\|_{L^q_\Omega} \to 0,$$

and

$$\left\|H_{f_2}^\Omega(g_m)\right\|_{L^q_\Omega} \lesssim \left(\int_{\mathbb{D}}|f_2|^q|g_m|^q\Omega dA\right)^{\frac{1}{q}} \to 0.$$

Consequently,

$$\lim_{m\to\infty}\|H_f^\Omega(g_m)\|_{L^q_\Omega} = 0.$$

This shows H_f^Ω is compact from A^p_ω to L^q_Ω. The proof is ended. □

Proof of Theorem 1.3. $(B) \Rightarrow (A)$ is trivial. We need only to prove the directions $(A) \Rightarrow (C)$, $(C) \Rightarrow (D)$ and $(D) \Rightarrow (B)$.

$(A) \Rightarrow (C)$. For $r \in (0, \alpha]$ fixed, take $\{z_j\}_{j=1}^\infty$ to be an $r/4$-lattice. It follows from the decomposition for A^p_ω (see [22, Proposition 14]) that for $\{\lambda_j\} \in l^p$, there holds

$$\left\|\sum_{j=1}^\infty \lambda_j b_{p,z_j}\right\|_{A^p_\omega} \leq C\|\{\lambda_j\}\|_{l^p}.$$

As in [11], take $\{\phi_j\}_{j=1}^\infty$ to be the sequence of Rademacher functions on $[0,1]$. Khintchine's inequality implies

$$\int_0^1\left|\sum_{j=1}^\infty \lambda_j\phi_j(t)b_{p,z_j}(z)\right|^q dt \simeq \left(\sum_{j=1}^\infty|\lambda_j|^2|b_{p,z_j}(z)|^2\right)^{\frac{q}{2}}.$$

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This together with Fubini's theorem gives

$$\begin{aligned}
&\int_0^1 \left\| H_f^\Omega \left(\sum_{j=1}^{\infty} \lambda_j \phi_j(t) b_{p,z_j} \right) \right\|_{L^q_\Omega}^q dt \\
\simeq\ & \sum_{k=1}^{\infty} \int_{D(z_k,r)} \left(\sum_{j=1}^{\infty} |\lambda_j|^2 |H_f^\Omega(b_{p,z_j})(z)|^2 \right)^{\frac{q}{2}} \Omega(z) dA(z) \\
\gtrsim\ & \sum_{k=1}^{\infty} |\lambda_k|^q b_{p,z_k}(z)^q \int_{D(z_k,r)} \left| f(z) - \frac{P_\Omega(f b_{p,z_k})}{b_{p,z_k}(z)}(z) \right|^q \Omega(z) dA(z) \\
\gtrsim\ & \sum_{k=1}^{\infty} |\lambda_k|^q b_{p,z_k}(z)^q \Omega(D(z_k,r)) G_{q,r}(f)(z_k)^q \\
\simeq\ & \sum_{k=1}^{\infty} |\lambda_k|^q \breve{\omega}(z_k)^{-\frac{q}{p}} \breve{\Omega}(z_k) G_{q,r}(f)(z_k)^q.
\end{aligned}$$

Meanwhile, the boundedness of $H_f^\Omega : A^p_\omega \to L^q_\Omega$ indicates

$$\begin{aligned}
\left\| H_f^\Omega \left(\sum_{j=1}^{\infty} \lambda_j \phi_j(t) b_{p,z_j} \right) \right\|_{L^q_\Omega} &\leq \|H_f^\Omega\|_{A^p_\omega \to L^q_\Omega} \cdot \left\| \sum_{j=1}^{\infty} \lambda_j \phi_j(t) b_{p,z_j} \right\|_{A^p_\omega} \\
&\lesssim \|H_f^\Omega\|_{A^p_\omega \to L^q_\Omega} \|\{|\lambda_j|^q\}\|_{l^{\frac{p}{q}}}^{\frac{1}{q}}.
\end{aligned}$$

Hence

$$\sum_{k=1}^{\infty} |\lambda_k|^q \breve{\omega}(z_k)^{-\frac{q}{p}} \breve{\Omega}(z_k) G_{q,r}(f)(z_k)^q \lesssim \|H_f^\Omega\|_{A^p_\omega \to L^q_\Omega}^q \|\{|\lambda_k|^q\}\|_{l^{\frac{p}{q}}}.$$

Since the conjugate of $\frac{p}{q}$ is $\frac{p}{p-q}$, a duality argument shows

$$\sum_{k=1}^{\infty} \breve{\omega}(z_k)^{-\frac{q}{p-q}} \breve{\Omega}(z_k)^{\frac{p}{p-q}} G_{q,r}(f)(z_k)^{\frac{pq}{p-q}} \lesssim \|H_f^\Omega\|_{A^p_\omega \to L^q_\Omega}^{\frac{pq}{p-q}}.$$

Notice that $\breve{\omega}(z) = (1-|z|)^2 \omega(z)$. For z and w with $\beta(z,\xi) < \frac{r}{2}$, one has $\breve{\omega}(z) \simeq \breve{\omega}(\xi)$ and

$$G_{q,\frac{r}{2}}(f)(\xi) \leq C G_{q,r}(f)(z). \tag{3.9}$$

Thus we have

$$\begin{aligned}
&\int_{\mathbb{D}} \Omega(\xi)^{\frac{p}{p-q}} \omega(\xi)^{-\frac{q}{p-q}} G_{q,\frac{r}{2}}(f)(\xi)^{\frac{pq}{p-q}} dA(\xi) \\
\leq\ & \sum_{k=1}^{\infty} \int_{D(z_k,\frac{r}{2})} \frac{1}{(1-|\xi|)^2} \breve{\omega}(\xi)^{-\frac{q}{p-q}} \breve{\Omega}(\xi)^{\frac{q}{p-q}} \breve{\Omega}(\xi) G_{q,r}(f)(\xi)^{\frac{pq}{p-q}} dA(\xi) \\
\lesssim\ & \sum_{k=1}^{\infty} G_{q,r}(f)(z_k)^{\frac{pq}{p-q}} \breve{\omega}(z_k)^{-\frac{q}{p-q}} \breve{\Omega}(z_k)^{\frac{q}{p-q}} \int_{D(z_k,\frac{r}{2})} \Omega(\xi) dA(\xi) \\
\simeq\ & \sum_{k=1}^{\infty} G_{q,r}(f)(z_k)^{\frac{pq}{p-q}} \breve{\omega}(z_k)^{-\frac{q}{p-q}} \breve{\Omega}(z_k)^{\frac{p}{p-q}}.
\end{aligned}$$

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This shows $\Omega^{\frac{1}{q}}\omega^{-\frac{1}{p}}G_{q,r}(f)\in L^{\frac{pq}{p-q}}(\mathbb{D})$ with

$$\left\|\Omega^{\frac{1}{q}}\omega^{-\frac{1}{p}}G_{q,r}(f)\right\|_{L^{\frac{pq}{p-q}}}\lesssim\left\|H_f^{\Omega}\right\|_{A^p_\omega\to L^q_\Omega}. \tag{3.10}$$

$(C)\Rightarrow(D)$. As in the proof of Theorem 1.1, set

$$f_1(z)=\textstyle\sum_{j=1}^{\infty}h_j(z)\phi_j(z)\in C^\infty(\mathbb{D})\text{ and }f_2=f-f_1.$$

By (3.9) we know

$$G_{q,\frac{r}{2}}(f)^{\frac{pq}{p-q}}(z_j)\lesssim\frac{1}{|D(z_j,r/2)|}\int_{D(z_j,r/2)}G_{q,r}(f)^{\frac{pq}{p-q}}(u)dA(u).$$

It follows from (3.3) that

$$\begin{aligned}\left[(1-|z|)\left|\overline{\partial}f_1(z)\right|\right]^{\frac{pq}{p-q}} &\lesssim \sum_{j\in J_z}G_{q,\frac{r}{2}}(f)^{\frac{pq}{p-q}}(z_j)\\ &\lesssim \frac{1}{\Omega(D(z,r))}\sum_{j\in J_z}\int_{D(z_j,r)}G_{q,r}(f)^{\frac{pq}{p-q}}(u)\Omega(u)dA(u)\\ &\lesssim \frac{1}{\Omega(D(z,r))}\int_{D(z,2r)}G_{q,r}(f)^{\frac{pq}{p-q}}(u)\Omega(u)dA(u).\end{aligned}$$

This and Fubini's theorem imply

$$\begin{aligned}&\int_{\mathbb{D}}\left[(1-|z|)\left|\overline{\partial}f_1(z)\right|\right]^{\frac{pq}{p-q}}\Omega(z)^{\frac{p}{p-q}}\omega(z)^{-\frac{q}{p-q}}dA(z)\\ \lesssim&\int_{\mathbb{D}}\frac{1}{\Omega(D(z,r))}\Omega(z)^{\frac{p}{p-q}}\omega(z)^{-\frac{q}{p-q}}dA(z)\int_{\mathbb{D}}\chi_{D(z,2r)}(u)G_{q,r}(f)(u)^{\frac{pq}{p-q}}\Omega(u)dA(u)\\ \simeq&\int_{\mathbb{D}}\Omega(u)^{\frac{p}{p-q}}\omega(u)^{-\frac{q}{p-q}}G_{q,r}(f)(u)^{\frac{pq}{p-q}}\left[\frac{1}{\Omega(D(u,r))}\int_{D(u,2r)}\Omega(z)dA(z)\right]dA(u)\\ \simeq&\int_{\mathbb{D}}\Omega(u)^{\frac{p}{p-q}}\omega(u)^{-\frac{q}{p-q}}G_{q,r}(f)(u)^{\frac{pq}{p-q}}dA(u).\end{aligned}$$

Thus we get (1.4) with

$$\left\|\Omega^{\frac{1}{q}}\omega^{-\frac{1}{p}}(1-|\cdot|)|\overline{\partial}f_1\right\|_{L^{\frac{pq}{p-q}}}\lesssim\left\|\Omega^{\frac{1}{q}}\omega^{-\frac{1}{p}}G_{q,r}(f)\right\|_{L^{\frac{pq}{p-q}}}. \tag{3.11}$$

For f_2, by (3.5) we obtain

$$\begin{aligned}M_r(|f_2|^q)^{\frac{1}{q}}(z) &\lesssim \sum_{j=1}^{\infty}\left(\frac{1}{\Omega(D(z,r))}\int_{D(z,r)\cap D(z_j,r/2)}|f-h_j|^q\,\Omega dA\right)^{\frac{1}{q}}\\ &\lesssim \frac{1}{\Omega(D(z,r))}\int_{D(z,2r)}G_{q,r}(f)(u)\Omega(u)dA(u)\\ &\lesssim \left(\frac{1}{\Omega(D(z,r))}\int_{D(z,2r)}G_{q,r}(f)(u)^{\frac{pq}{p-q}}\Omega(u)dA(u)\right)^{\frac{p-q}{pq}}.\end{aligned}$$

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This together with (3.9) shows

$$\begin{aligned}&\left\|\Omega^{\frac{1}{q}}\omega^{-\frac{1}{p}}M_r(|f_2|^q)^{\frac{1}{q}}\right\|_{L^{\frac{pq}{p-q}}}\\ \lesssim\ &\int_{\mathbb{D}}\Omega(z)^{\frac{p}{p-q}}\omega(z)^{-\frac{q}{p-q}}\left(\frac{1}{\Omega(D(z,r))}\int_{D(z,2r)}G_{q,r}(f)(u)^{\frac{pq}{p-q}}\Omega(u)dA(u)\right)dA(z)\\ \lesssim\ &\int_{\mathbb{D}}\Omega(z)^{\frac{p}{p-q}}\omega(z)^{-\frac{q}{p-q}}G_{q,2r}(f)(z)^{\frac{pq}{p-q}}dA(z).\end{aligned}$$

This shows

$$\left\|\Omega^{\frac{1}{q}}\omega^{-\frac{1}{p}}M_r(|f_2|^q)^{\frac{1}{q}}\right\|_{L^{\frac{pq}{p-q}}}\lesssim\left\|\Omega^{\frac{1}{q}}\omega^{-\frac{1}{p}}G_{q,2r}(f)\right\|_{L^{\frac{pq}{p-q}}}. \tag{3.12}$$

It is trivial that the condition (1.5) is independent of r, these give the statement (D).

Now we prove $(D)\Rightarrow(B)$. Set $d\nu=\left[(1-|\cdot|)|\overline{\partial}f_1|\right]^q\Omega dA$. It follows from Hölder's inequality with $\frac{p}{p-q}$ and its conjugate to get

$$\begin{aligned}&\int_{D(\xi,r)}\left[(1-|\zeta|)|\overline{\partial}f_1(\zeta)|\right]^q\Omega(\zeta)dA(\zeta)\\ \lesssim\ &\left\{\int_{D(\xi,r)}\left[(1-|\zeta|)^q|\overline{\partial}f_1(\zeta)|^q\Omega(\zeta)^{\frac{p-q}{p}}\right]^{\frac{p}{p-q}}dA(\zeta)\right\}^{\frac{p-q}{p}}\cdot\left(\int_{D(\xi,r)}\Omega(\zeta)dA(\zeta)\right)^{\frac{q}{p}}\\ \simeq\ &\breve{\Omega}(\xi)^{\frac{q}{p}}\left[\int_{D(\xi,r)}(1-|\zeta|)^{\frac{pq}{p-q}}|\overline{\partial}f_1(\zeta)|^{\frac{pq}{p-q}}\Omega(\zeta)dA(\zeta)\right]^{\frac{p-q}{p}}\end{aligned}$$

This, together with Fubini's theorem, gives

$$\begin{aligned}&\left\|\frac{\nu(D(\cdot,r))}{\omega(D(\cdot,r))}\right\|_{L_\omega^{\frac{p}{p-q}}}\\ =\ &\int_{\mathbb{D}}\left[\frac{\int_{D(\xi,r)}(1-|\zeta|)^q|\overline{\partial}f_1(\zeta)|^q\Omega(\zeta)dA(\zeta)}{\omega(D(\xi,r))}\right]^{\frac{p}{p-q}}\omega(\xi)dA(\xi)\\ \lesssim\ &\int_{\mathbb{D}}\left(\int_{D(\xi,r)}(1-|\zeta|)^{\frac{pq}{p-q}}|\overline{\partial}f_1(\zeta)|^{\frac{pq}{p-q}}\Omega(\zeta)dA(\zeta)\right)\frac{\breve{\Omega}(\xi)^{\frac{q}{p-q}}}{\breve{\omega}(\xi)^{\frac{p}{p-q}}}\omega(\xi)dA(\xi)\\ =\ &\int_{\mathbb{D}}\left(\int_{D(\zeta,r)}\frac{\breve{\Omega}(\xi)^{\frac{q}{p-q}}}{\breve{\omega}(\xi)^{\frac{p}{p-q}}}\omega(\xi)dA(\xi)\right)(1-|\zeta|)^{\frac{pq}{p-q}}|\overline{\partial}f_1(\zeta)|^{\frac{pq}{p-q}}\Omega(\zeta)dA(\zeta).\end{aligned}$$

Since $\breve{\Omega}(\xi)\simeq\breve{\Omega}(\zeta)$ for $\xi\in D(\zeta,r)$, we have

$$\int_{D(\zeta,r)}\frac{\breve{\Omega}(\xi)^{\frac{q}{p-q}}}{\breve{\omega}(\xi)^{\frac{p}{p-q}}}\omega(\xi)dA(\xi)\simeq\frac{\breve{\Omega}(\zeta)^{\frac{q}{p-q}}}{\breve{\omega}(\zeta)^{\frac{p}{p-q}}}\int_{D(\zeta,r)}\omega(\xi)dA(\xi)\simeq\frac{\Omega(\zeta)^{\frac{q}{p-q}}}{\omega(\zeta)^{\frac{q}{p-q}}}.$$

Therefore,

$$\left\|\frac{\nu(D(\cdot,r))}{\omega(D(\cdot,r))}\right\|_{L_\omega^{\frac{p}{p-q}}}\lesssim\int_{\mathbb{D}}\left[\Omega(\zeta)^{\frac{1}{q}}\omega(\zeta)^{-\frac{1}{p}}(1-|\zeta|)|\overline{\partial}f_1(\zeta)|\right]^{\frac{pq}{p-q}}dA(\zeta).$$

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By (1.4) we know $d\nu$ is a q-Carleson measure for A^p_ω. Lemma 2.3 tells us $Id : A^p_\omega \hookrightarrow L^q(d\nu)$ is compact and

$$\|Id\|^q_{A^p_\omega \hookrightarrow L^q(d\nu)} \lesssim \left\| \frac{\nu(D(\cdot,r))}{\omega(D(\cdot,r))} \right\|_{L_\omega^{\frac{p}{p-q}}} \lesssim \left\| \Omega^{\frac{1}{q}} \omega^{-\frac{1}{p}} (1-|\cdot|) |\overline{\partial} f_1| \right\|^q_{L^{\frac{pq}{p-q}}} .$$

Now for $g \in H^\infty$, which is dense in A^p_ω, we have

$$\begin{aligned} \|H^\Omega_{f_1} g\|_{L^q_\Omega} &\lesssim \|(1-|\cdot|) g \overline{\partial} f_1\|_{L^q_\Omega} \lesssim \|Id\|_{A^p_\omega \to L^q(d\nu)} \|g\|_{L^p_\omega} \\ &\lesssim \left\| \Omega^{\frac{1}{q}} \omega^{-\frac{1}{p}} (1-|\cdot|) |\overline{\partial} f_1| \right\|_{L^{\frac{pq}{p-q}}} \|g\|_{L^p_\omega}. \end{aligned}$$

Therefore, $H^\Omega_{f_1} : A^p_\omega \to L^q_\Omega$ is bounded with the norm estimate

$$\|H^\Omega_{f_1}\|_{A^p_\omega \to L^q_\Omega} \lesssim \left\| \Omega^{\frac{1}{q}} \omega^{-\frac{1}{p}} (1-|\cdot|) |\overline{\partial} f_1| \right\|_{L^{\frac{pq}{p-q}}} . \tag{3.13}$$

We claim $H^\Omega_{f_1}$ is compact as well. To see this, let $\{g_m\}^\infty_{m=1}$ be any bounded sequence in A^p_ω with $g_m \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $m \to \infty$, we need only to prove $H^\Omega_{f_1}(g_m) \to 0$ in L^q_Ω. For this purpose, for each m, pick some $t_m \in (1-1/m, 1)$ so that $\|g_m(\cdot) - g_m(t_m \cdot)\|_{A^p_\omega} < \frac{1}{m}$. Set $h_m(\cdot) = g_m(t_m \cdot) \in H^\infty$. Since ω is radial, $\|h_m\|_{A^p_\omega} \le \|g_m\|_{A^p_\omega}$. As in Lemma 2.5, set

$$u_m(z) = \sum_{j=1}^\infty B_{z_j}(z) \int_{\mathbb{D}} \frac{\phi_j(\xi)}{(\xi - z) B_{z_j}(\xi)} h_m(\xi) \overline{\partial} f_1(\xi) dA(\xi).$$

Then, $\overline{\partial} u_m = h_m \overline{\partial} f_1$ and

$$\|u_m\|_{L^q_\Omega} \lesssim \|(h_m \overline{\partial} f_1)\|_{L^q_{(1-|\cdot|)^q \Omega}} \simeq \|h_m\|^q_{L^q(d\nu)}.$$

It follows from the compactness of $Id : A^p_\omega \hookrightarrow L^q(d\nu)$ that $\lim_{m\to\infty} \|h_m\|^q_{L^q(d\nu)} = 0$. Then $\lim_{m\to\infty} \|u_m\|_{L^q_\Omega} = 0$. Therefore,

$$\lim_{m\to\infty} \left\| H^\Omega_{f_1}(h_m) \right\|_{L^q_\Omega} \le (1 + \|P_\Omega\|_{L^q_\Omega}) \lim_{m\to\infty} \|u_m\|_{L^q_\Omega} = 0.$$

Meanwhile, we have

$$\lim_{m\to\infty} \left\| H^\Omega_{f_1}(g_m - h_m) \right\|_{L^q_\Omega} \le \|H^\Omega_{f_1}\|_{A^p_\omega \to L^q_\Omega} \lim_{m\to\infty} \|g_m - h_m\|_{A^p_\omega} = 0.$$

Therefore,

$$\lim_{m\to\infty} \left\| H^\Omega_{f_1}(g_m) \right\|_{L^q_\Omega} \le \lim_{m\to\infty} \left\{ \left\| H^\Omega_{f_1}(g_m - h_m) \right\|_{L^q_\Omega} + \left\| H^\Omega_{f_1}(h_m) \right\|_{L^q_\Omega} \right\} = 0,$$

which gives the compactness of $H^\Omega_{f_1}$ from A^p_ω to L^q_Ω.

For f_2 satisfying (1.5), $d\mu = |f_2|^q \Omega dA$ is a vanishing q-Carleson measure for A^p_ω. Equivalently, $Id : A^p_\omega \to L^q(d\mu)$ is compact. By

$$\left\| H^\Omega_{f_2}(g) \right\|_{L^q_\Omega} \lesssim \|f_2 g\|_{L^q_\Omega} = \|Id(g)\|_{L^q_{|f_2|^q \Omega}} \tag{3.14}$$

we know $H^\Omega_{f_2}$ is compact from A^p_ω to L^q_Ω as well. This finishes the proof of implication $(D) \Rightarrow (B)$.

Furthermore, from (3.13), (3.14) and (3.11), (3.12) we have

$$\|H^\Omega_f\|_{A^p_\omega \to L^q_\Omega} \lesssim \inf \left\{ \|H^\Omega_{f_1}\|_{A^p_\omega \to L^q_\Omega} + \|H^\Omega_{f-2}\|_{A^p_\omega \to L^q_\Omega} \right\} \lesssim \left\| \Omega^{\frac{1}{q}} \omega^{-\frac{1}{p}} G_{q,r}(f) \right\|_{L^{\frac{pq}{p-q}}} ,$$

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where the "inf" is taken over all decomposition $f = f_1 + f_2$ as (1.4). This and (3.10) imply (1.6). The proof is finished.

Recall that for $f \in L^p_{loc}$ and $r > 0$,

$$MO_{p,r}(f)(z) = \left\{ \frac{1}{|D(z,r)|} \int_{D(z,r)} |f - f_{D(z,r)}|^q dA \right\}^{\frac{1}{p}},$$

where $f_{D(z,r)} = \frac{1}{|D(z,r)|} \int_{D(z,r)} f dA$. And

$$\mathrm{Oss}_r(f)(z) = \sup_{\xi \in D(z,r)} |f(\xi) - f(z)|.$$

□

As an application of the main results, we can characterize real-valued functions $f \in L^1_\Omega$ such that H^Ω_f is bounded (or compact) from A^p_ω to L^q_Ω with $1 < p, q < \infty$. This is equivalent to show complex-valued functions $f \in L^1_\Omega$ such that both $H^\Omega_f, H^\Omega_{\overline{f}}$ are bounded (or compact) between the above spaces. The proofs of the following corollaries are similar to that of Theorems 4.5-4.7 in [10], so we omit the details here.

Corollary 3.1. *Suppose $\omega, \Omega \in \mathcal{R}$, $1 < p \le q < \infty$. Then for $f \in L^1_\Omega$,*

(1) *Both $H^\Omega_f, H^\Omega_{\overline{f}} : A^p_\omega \to L^q_\Omega$ are bounded if and only if for some (or any) $r > 0$, $\breve{\Omega}^{\frac{1}{q}} \breve{\omega}^{-\frac{1}{p}} MO_{q,r}(f) \in L^\infty$, as well as equivalent to $f = f_1 + f_2$ with $f_1 \in C^1(\mathbb{D})$, and for some (or any) $r > 0$,*

$$\breve{\Omega}^{\frac{1}{q}} \breve{\omega}^{-\frac{1}{p}} \cdot Oss_r(f_1) \in L^\infty \quad \text{and} \quad \breve{\Omega}^{\frac{1}{q}} \breve{\omega}^{-\frac{1}{p}} M_r(|f_2|^q)^{\frac{1}{q}} \in L^\infty.$$

Furthermore,

$$\left\| H^\Omega_f \right\|_{A^p_\omega \to L^q_\Omega} + \left\| H^\Omega_{\overline{f}} \right\|_{A^p_\omega \to L^q_\Omega} \simeq \left\| \breve{\Omega}^{\frac{1}{q}} \breve{\omega}^{-\frac{1}{p}} MO_{q,r}(f) \right\|_{L^\infty}.$$

(2) *$H^\Omega_f, H^\Omega_{\overline{f}} : A^p_\omega \to L^q_\Omega$ are simultaneously compact if and only if for some (or any) $r > 0$, $\lim_{|z|\to 1} \breve{\Omega}(z)^{\frac{1}{q}} \breve{\omega}(z)^{-\frac{1}{p}} MO_{q,r}(f)(z) = 0$, which is also equivalent to $f = f_1 + f_2$ with $f_1 \in C^1(\mathbb{D})$ satisfying*

$$\lim_{|z|\to 1} \breve{\Omega}(z)^{\frac{1}{q}} \breve{\omega}(z)^{-\frac{1}{p}} Oss_r(f_1)(z) = 0,$$

and

$$\lim_{|z|\to 1} \breve{\Omega}(z)^{\frac{1}{q}} \breve{\omega}(z)^{-\frac{1}{p}} M_r(|f_2|^q)^{\frac{1}{q}}(z) = 0$$

for some (or any) $r > 0$.

Corollary 3.2. *Suppose $\omega, \Omega \in \mathcal{R}$, $1 < q < p < \infty$. Then for $f \in L^1_\Omega$, the following statements are equivalent:*

(A) *$H^\Omega_f, H^\Omega_{\overline{f}} : A^p_\omega \to L^q_\Omega$ are bounded;*

(B) *$H^\Omega_f, H^\Omega_{\overline{f}} : A^p_\omega \to L^q_\Omega$ are compact;*

(C) *For some (or any) $r > 0$, $\Omega^{\frac{1}{q}} \omega^{-\frac{1}{p}} MO_{q,r}(f) \in L^{\frac{pq}{p-q}}$;*

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(D) *$f = f_1 + f_2$ with $f_1 \in C^1(\mathbb{D})$, for some (or any) $r > 0$ there hold*

$$\Omega^{\frac{1}{q}}\omega^{-\frac{1}{p}}Oss_r(f_1) \in L^{\frac{pq}{p-q}},$$

and

$$\Omega^{\frac{1}{q}}\omega^{-\frac{1}{p}}M_r(|f_2|^q)^{\frac{1}{q}} \in L^{\frac{pq}{p-q}}.$$

Furthermore,

$$\left\|H_f^\Omega\right\|_{A^p_\omega \to L^q_\Omega} + \left\|H_{\overline{f}}^\Omega\right\|_{A^p_\omega \to L^q_\Omega} \simeq \left\|\breve{\Omega}^{\frac{1}{q}}\breve{\omega}^{-\frac{1}{p}}MO_{q,r}(f)\right\|_{L^{\frac{pq}{p-q}}}.$$

Remark 3.3. Given $1 < q < p < \infty$ and $\alpha, \beta > -1$, Lv and Zhu in [14] obtained the equivalent conditions for H_f^β and $H_{\overline{f}}^\beta$ are both bounded (or compact) from $A^p_{(1-|\cdot|^2)^\alpha}$ to $L^q_{(1-|\cdot|^2)^\beta}$ under the restriction "$pn < q(n + 1 + \alpha)$". They were not sure if this restriction could remove. From Corollary 3.2 we know, the condition is not necessary at least in the case $n = 1$.

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