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SOME CAYLEY GRAPHS WITH PROPAGATION TIME 1

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Abstract. In this paper we study the zero forcing number as well as the propagation time of Cayley graph $Cay(G, \Omega)$, where G is a finite group and $\Omega \subset G \setminus \{1\}$ is an inverse closed generator set of G. It is proved that the propagation time of $Cay(G, \Omega)$ is 1 for some Cayley graphs on dihedral groups and finite cyclic groups with special generator set Ω.

1. Introduction

In this paper, all graphs are assumed to be finite, simple and undirected. We will often use the notation $\Gamma = (V, E)$ to denote the graph with non-empty vertex set $V = V(\Gamma)$ and edge set $E = E(\Gamma)$. *Order* of a graph is the number of vertices in the graph and *size* of a graph is the number of edges in the graph. An edge of Γ with endpoints u and v is denoted by $u - v$. For every vertex $x \in V(\Gamma)$, the *open neighborhood* of vertex x is denoted by $N(x)$ and defined as $N(x) = \{y \in V(\Gamma) \mid x - y\}$. Also the *close neighborhood* of vertex $x \in V(\Gamma)$, $N[x]$, is $N[x] = N(x) \cup \{x\}$. The *degree* of a vertex $x \in V(\Gamma)$ is $\deg_\Gamma(x) = \big|N(x)\big|$. The *minimum degree* and *maximum degree* of a graph Γ denoted by $\delta(\Gamma)$ and $\Delta(\Gamma)$, respectively. The *complement* of graph Γ denoted by $\overline{\Gamma}$ is a graph with vertex set $V(\Gamma)$ which $e \in E(\overline{\Gamma})$ if and only if $e \notin E(\Gamma)$. For any $S \subseteq V(\Gamma)$, the *induced subgraph* on S, denoted by $\Gamma[S]$ is the subgraph whose vertex set is S and which contains all edges with both endpoints in S. The set $S \subseteq V(\Gamma)$, is independent, if $\Gamma[S]$ is empty graph.

A $t-$*partite graph* is a graph whose vertices are or can be partitioned into t different independent

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sets. A *complete* $t-$*partite graph* is a $t-$partite graph in which there is an edge between every pair of vertices from different independent sets. A *complete multipartite graph* is a complete $t-$partite graph for some t.

Let G be a non-trivial group with identity element 1 and $\Omega \subseteq G$ such that $1 \notin \Omega$, $\Omega = \Omega^{-1} = \{\omega^{-1} \mid \omega \in \Omega\}$. The *Cayley graph* of G and Ω, denoted by $Cay(G, \Omega)$, is a graph with vertex set G and two vertices u and v are adjacent if and only if $uv^{-1} \in \Omega$.

The set of $n \times n$ real symmetric matrices will be denoted by $S_n(\mathbb{R})$. For $A \in S_n(\mathbb{R})$, the graph of $A = (a_{ij})$, denoted by $\mathcal{G}(A)$, is a graph with vertices $\{1, \dots, n\}$ and edges $\{i - j \mid a_{ij} \neq 0, 1 \leqslant i, j \leqslant n\}$. Note that the diagonal of A is ignored in determining $\mathcal{G}(A)$.
The set of symmetric matrices of graph Γ is defined by

$$S(\Gamma) = \{A \in S_n(\mathbb{R}) \mid \mathcal{G}(A) = \Gamma\}.$$

The maximum nullity of Γ is

$$M(\Gamma) = \max\{null(A) \mid A \in S(\Gamma)\}$$

and the minimum rank of G is

$$mr(\Gamma) = \min\{rank(A) \mid A \in S(\Gamma)\}.$$

A *matching* in a graph is a set of edges without common vertices. A *perfect matching* of graph is a matching in which every vertex of the graph is incident to exactly one edge of the matching. Suppose that $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ are two graphs of equal order and $\mu : V_1 \to V_2$ is a bijection. Define the *matching graph* (H_1, H_2, μ) to be the graph constructed with the disjoint union of H_1, H_2 and perfect matching between V_1 and V_2 defined by μ.

Let each vertex of a graph Γ be given one of two colors "black" and "white". Let Z denote the (initial) set of black vertices in Γ. If a white vertex u_2 is the only white neighbor of a black vertex u_1, then u_1 changes the color of u_2 to black (*color-change rule*) and we say "u_1 forces u_2". The set Z is said to be a *zero forcing* set of Γ if all of the vertices of Γ will be turned black after finitely many applications of the color-change rule. The *zero forcing number* of Γ, $Z(\Gamma)$, is the minimum cardinality among all zero forcing sets. The notation of a zero forcing set, as well as the associated zero forcing number, of a simple graph was introduced by the "AIM Minimum Rank-Special Graphs Work Group" in (2008) [2]. They used the technique of zero forcing parameter of graph Γ and found an upper bound for the maximum nullity of Γ related to zero forcing sets. For more results, see [3, 4, 6], [7] and [12].
Let $\Gamma = (V, E)$ be a graph and Z a zero forcing set of Γ. Define $Z^{(0)} = Z$ and for $t \geqslant 0$, $Z^{(t+1)}$ is the set of vertices w for which there exists a vertex $b \in \bigcup_{s=0}^{t} Z^{(s)}$ such that w is the only neighbor of b not in $\bigcup_{s=0}^{t} Z^{(s)}$. The propagation time of Z in Γ, denoted by $Pt(\Gamma, Z)$, is the smallest integer t_0 such that $V = \bigcup_{t=0}^{t_0} Z^{(t)}$. The *propagation time* of Γ is

$$Pt(\Gamma) = \min\{Pt(\Gamma, Z) \mid Z \; is \; a \; minimum \; zero \; forcing \; set \; of \; \Gamma\}.$$

The propagation time of a zero forcing set was implicit in [5] and explicit in [10]. In 2012 Hogben et al. [8] established some results regarding graphs having propagation time 1.
These motivated us to consider the zero forcing number and propagation time of some Cayley graphs.

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We show that $Pt(Cay(G,\Omega)) = 1$ for some Cayley graphs on dihedral groups and finite cyclic groups with special generator set Ω.

2. Preliminary

For investigating the zero forcing number and propagation time of graphs, the following Lemmas and Theorems are useful.

Theorem 2.1 ([4]). *For any graph Γ, $\delta(\Gamma) \leqslant Z(\Gamma)$, where $\delta(\Gamma)$ is the minimum degree of the graph Γ.*

Theorem 2.2 ([7]). *Let Γ be a connected graph of order $n \geqslant 2$. Then $Z(\Gamma) = n - 1$ if and only if Γ is isomorphic to a complete graph of order n.*

Theorem 2.3 ([2]). *Let $\Gamma = (V, E)$ be a graph and $Z \subseteq V$ a zero forcing set for Γ. Then $M(\Gamma) \leqslant Z(\Gamma)$.*

Lemma 2.4 ([2]). *Let λ be an eigenvalue of the adjacency matrix of graph Γ with multiplicity of m. Then $M(\Gamma) \geqslant m$.*

Theorem 2.5 ([8]). *Let Γ be a graph. Then any two of the following conditions imply the third:*
1. $|\Gamma| = 2Z(\Gamma)$.
2. $Pt(\Gamma) = 1$.
3. Γ is a matching graph.

Theorem 2.6 ([11]). *Let $t \geq 2$ and $K_{n_1,\cdots,n_t}$ be a complete multipartite graph, with at least one i $(1 \leqslant i \leqslant t)$ such that $n_i > 1$. Then $Z(K_{n_1,\cdots,n_t}) = n_1 + \cdots + n_t - 2$.*

Lemma 2.7. *Let $t \geq 2$ and $K_{n_1,\cdots,n_t}$ $(n_1 \leqslant n_2 \leqslant \cdots \leqslant n_t)$ be a complete multipartite graph. If $1 = n_1 = n_2 = \cdots = n_{t-1} < n_t$, then $Pt(K_{n_1,\cdots,n_t}) = 2$. Otherwise, $Pt(K_{n_1,\cdots,n_t}) = 1$.*

Proof. Let $V(K_{n_1,\cdots,n_t})$ is partitioned parts of $V_1, \cdots, V_t$ with $|V_i| = n_i$ for $1 \leq i \leq t$. Let Z be a zero forcing set of $K_{n_1,\cdots,n_t}$ with minimum cardinality. By Theorem 2.6, $|Z| = n_1 + \cdots + n_t - 2$. Also let $V(K_{n_1,\cdots,n_t}) \setminus Z = \{x, y\}$.
If $1 = n_1 = \cdots = n_{t-1} < n_t$, then $x \in V_t$ and $y \in V_i$ for some $1 \leqslant i \leqslant t-1$. Since y is not black vertex, x can not be forced by any black vertices in the first stage. But every black vertex in V_t forces y and then x is forced by y. Therefore, $Z^{(0)} = Z$, $Z^{(1)} = \{y\}$ and $Z^{(2)} = \{x\}$. Hence $Pt(K_{n_1,\cdots,n_t}, Z) = 2$ and so $Pt(K_{n_1,\cdots,n_t}) = 2$.
Let there are i and j with $1 \leqslant i < j \leqslant t$ such that $1 < |V_i|$ and $1 < |V_j|$, $x \in V_i$ and $y \in V_j$. Then $Z = V(K_{n_1,\cdots,n_t}) \setminus \{x, y\}$ is a zero forcing set of $K_{n_1,\cdots,n_t}$. Furthermore every black vertex in V_i forces y and every black vertex in V_j forces x, simultaneously. Thus, $Z^{(0)} = Z$, $Z^{(1)} = \{x, y\}$ and $V(K_{n_1,\cdots,n_t}) = Z^{(0)} \cup Z^{(1)}$. Therefore $Pt(K_{n_1,\cdots,n_t}, Z) = 1$ and so $Pt(K_{n_1,\cdots,n_t}) = 1$. □

Lemma 2.8 ([9]). *Let G be a group and H be a proper subgroup of G. Also let $[G : H] = t$. If $\Omega = G \setminus H$, then $Cay(G, \Omega)$ is a complete t-partite graph.*

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Theorem 2.9 ([11]). *Let $G = D_{2n} = \langle a, b \mid a^n = b^2 = (ab)^2 = 1\rangle$ be the dihedral group of order $2n$, where $n = 2k$. Also let $\Omega = \{a, a^3, \cdots, a^{2k-1}, b\}$. Then $Z(Cay(D_{2n}, \Omega)) = 2|\Omega| - 2$.*

A graph is called *integral, if its adjacency eigenvalues are integers.*

Theorem 2.10 ([1]). *Let $T_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1}\rangle$, $n = 2m+1$ where $m \in \mathbb{N}$ and $\Omega = \{a^k \mid 1 \leqslant k \leqslant 2n-1, k \neq n\} \cup \{ab, a^{n+1}b\}$. Then $Cay(T_{4n}, \Omega)$ is integral.*

Lemma 2.11. *Let $T_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1}\rangle, n = 2m+1$ where $m \in \mathbb{N}$ and $\Omega = \{a^k \mid 1 \leqslant k \leqslant 2n-1, k \neq n\} \cup \{ab, a^{n+1}b\}$. Then $Z(Cay(T_{4n}, \Omega)) \leqslant 3n$. If $n = 3$, then $Z(Cay(T_{12}, \Omega)) \leqslant 8$.*

Proof. Let $n = 3$ and $X = \{1, a, a^2, a^4, a^5, b, ab, a^2b\}$ be the set of initial black vertices of $Cay(T_{12}, \Omega)$. Then since $N(1) = \Omega$, and a^4b is the only white neighbor of 1, 1 forces a^4b. We have $N(b) = \{a, a^4, ab, a^2b, a^4b, a^5b\}$. So a^5b is the only white neighbor of b. Hence, b forces a^5b. Since $N(a^2) = \{1, a, a^3, a^4, a^2b, a^5b\}$ and a^3 is the only white neighbor of a^2, a^3 is forced by a^2. Finally, a^3b is forced by vertex a. Thus X is a zero forcing set of $Cay(T_{12}, \Omega)$ and so $Z(Cay(T_{12}, \Omega)) \leqslant 8$.
Now, let $n > 3$ and $X = \langle a\rangle \cup \{a^ib \mid 0 \leqslant i \leqslant n-1\}$ be the set of initial black vertices of $Cay(T_{4n}, \Omega)$. For every $k \in \{0, 1, \cdots, n-1\}$, we have $N(a^k) = \{a^{n-k+1}b, a^{2n-k+1}b\} \cup \langle a\rangle \setminus \{a^k, a^{n+k}\}$. Thus $a^{n+1}b$ and a^nb are the only white neighbors of vertices 1 and a, respectively. Also $a^{2n-k+1}b$ is the only white neighbor of vertex a^k, for $2 \leqslant k \leqslant n-1$. Hence, 1 forces $a^{n+1}b$, a forces a^nb and a^k forces $a^{2n-k+1}b$, for $2 \leqslant k \leqslant n-1$. Thus X is a zero forcing set of $Cay(T_{4n}, \Omega)$. Therefore $Z(Cay(T_{4n}, \Omega)) \leqslant 3n$. □

3. Main results

Let G be a finite group and $\Omega = \Omega^{-1} \subset G \setminus \{1\}$ be a generator of G. In the following results we provide groups G and sets Ω such that $Pt(Cay(G, \Omega)) = 1$.

Theorem 3.1. *Let G be a finite group and $G = \langle\Omega\rangle$, where $\Omega = G \setminus \{1, a\}$ and $o(a) = 2$. Then $Pt(Cay(G, \Omega)) = 1$.*

Proof. Since $\{1, a\}$ is a subgroup of G, by Lemma 2.8, $Cay(G, \Omega)$ is a complete multipartite graph with more than one part of order at least two. By Lemma 2.7, $Pt(Cay(G, \Omega)) = 1$. □

Theorem 3.2. *Let G be a finite group of order n and $G = \langle\Omega\rangle$, where $\Omega = G \setminus \{1, a, b\}$ and $\Omega = \Omega^{-1}$. Then $Pt(Cay(G, \Omega)) = 1$ if and only if one of the followings hold:*
1. $o(a) \in \{3, 4, 6\}$.
2. $o(a) = 2$ and $ab = ba$.
3. $o(a) = 2, ab \neq ba$ and $o(ab) = 3$.

Proof. ($\Rightarrow$) Let $Pt(Cay(G, \Omega)) = 1$ and on the contrary $o(a) = 5$ or $o(a) \geqslant 7$ or $o(a) = 2$, $ab \neq ba$ and $o(ab) \neq 3$.
Let $Pt(Cay(G, \Omega)) = 1$ and $o(a) = 5$. Then $N(a) = G \setminus \{1, a^2\}$ and $N(a^{-1}) = G \setminus \{1, a^{-2}\}$. With a not so difficult calculation we have $X = G \setminus \{a^3, a, a^4\}$ is a zero forcing set of $Cay(G, \Omega)$. By Theorem

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2.1, $Z(Cay(G,\Omega)) = n-3$. Let B be a zero forcing set of $Cay(G,\Omega)$ with minimum cardinality such that $Pt(Cay(G,\Omega),B) = 1$. Since $Cay(G,\Omega)$ is a vertex transitive graph, we may assume that $1 \in B$ is a first forcing vertex. Hence $\{a,a^4\} \cap B = \emptyset$. So $B = G \setminus \{x,a,a^4\}$, where $x \in \Omega$.
If $x \neq a^2$ and $x \neq a^3$, then $x \in N(a^2) \cap N(a^3)$. Thus $B^{(1)} = \{x\}$, $B^{(2)} = \{a,a^4\}$ and so $G = B^{(0)} \cup B^{(1)} \cup B^{(2)}$, where $B^{(0)} = B$. Hence $Pt(Cay(G,\Omega),B) = 2$.
If $x = a^2$, then $B^{(1)} = \{a,a^2\}$, $B^{(2)} = \{a^4\}$. So $G = B^{(0)} \cup B^{(1)} \cup B^{(2)}$, where $B^{(0)} = B$. Thus $Pt(Cay(G,\Omega),B) = 2$.
If $x = a^3$, then $B^{(1)} = \{a^3,a^4\}$, $B^{(2)} = \{a\}$. So $G = B^{(0)} \cup B^{(1)} \cup B^{(2)}$, where $B^{(0)} = B$. Thus $Pt(Cay(G,\Omega),B) = 2$. Which is not true.

Let $Pt(Cay(G,\Omega)) = 1$ and $o(a) \geqslant 7$. Then $N(a) = G \setminus \{1,a^2\}$ and $N(a^{-1}) = G \setminus \{1,a^{-2}\}$. It is easy to see that $X = G \setminus \{a^3,a,a^{-1}\}$ is a zero forcing set of $Cay(G,\Omega)$. By Theorem 2.1, $Z(Cay(G,\Omega)) = n-3$. Let B be a zero forcing set of $Cay(G,\Omega)$ with minimum cardinality and $Pt(Cay(G,\Omega),B) = 1$. Since $Cay(G,\Omega)$ is a vertex transitive graph, we may assume that $1 \in B$ is a first forcing vertex. Hence $B = G\setminus\{x,a,a^{-1}\}$, where $x \in \Omega$. It is clear that $a^2 \in N(a^{-2})$, $a^{-1} \in N(a^2)$ and $a \in N(a^{-2})$. If $x = a^2$ or $x = a^{-2}$, then $B^{(1)} = \{a^2\}$ or $B^{(1)} = \{a^{-2}\}$, respectively. However $G \neq B^{(0)} \cup B^{(1)}$ and so $Pt(Cay(G,\Omega),B) \geq 2$. This is a contradiction. Now let $x \in \Omega \setminus \{a^2,a^{-2}\}$. Since $Pt(Cay(G,\Omega),B) = 1$, $x \notin N(a^2) \cup N(a^{-2})$. Hence $a^3 = x = a^{-3}$. Thus $o(a) = 6$, which is a contradiction.

Let $Pt(Cay(G,\Omega)) = 1$, $o(a) = 2$, $ab \neq ba$ and $o(ab) \neq 3$. If $o(ab) = 2$, then $ab = ba$, which is not true. So $o(ab) \geq 4$. Since a is not adjacent to ba and b is not adjacent to ab, $Z = G \setminus \{a,b,ab\}$ is a zero forcing set of $Cay(G,\Omega)$ and so $Z(Cay(G,\Omega)) \leq n-3$. By Theorem 2.1, $Z(Cay(G,\Omega)) \geq n-3$. Hence $Z(Cay(G,\Omega)) = n-3$. Since $Pt(Cay(G,\Omega)) = 1$, we may assume that B is a zero forcing set of $Cay(G,\Omega)$ with $Pt(Cay(G,\Omega),B) = 1$ and $1 \in B$ is a first forcing vertex. Thus $B = G \setminus \{a,b,x\}$, where $x \in \Omega$. Hence there are two elements x' and x'' in Ω such that x is not adjacent to x' and x''. Furthermore, x' is not adjacent to b and x'' is not adjacent to a. By easy computing we have $x' = ab$ and $x'' = ba$ and $aba = x = bab$. Thus $ababab = 1$, which is false.

($\Leftarrow$) Conversely, let $o(a) \in \{3,4,6\}$ or $o(a) = 2$ and $ab = ba$ or $o(a) = 2, ab \neq ba$ and $o(ab) = 3$.
If $o(a) = 3$, then $\{1,a,b\}$ is a subgroup of G. By Lemma 2.8, $Cay(G,\Omega)$ is a complete multipartite graph. Hence, $Pt(Cay(G,\Omega)) = 1$, by Lemma 2.7.
Let $o(a) = 4$ and Z be a zero forcing set of $Cay(G,\Omega)$ with $Z(Cay(G,\Omega)) = |Z|$. Since $Cay(G,\Omega)$ is a vertex transitive graph, we may assume that $1 \in Z$ is a first forcing vertex. Since $|\Omega| = n-3$ and $N(1) = \Omega$, there is $C \subseteq \Omega \cap Z$ such that $|C| = n-4$. Hence, $n-3 \leqslant |Z|$. Also we have $N[a] = N[b] = G \setminus \{1,a^2\}$. Thus $a \in Z$ or $b \in Z$. So $n-2 \leqslant |Z|$. Since $Cay(G,\Omega)$ is not a complete graph, by Lemma 2.2, $Z(Cay(G,\Omega)) = n-2$. It is clear that $B = G \setminus \{a,a^2\}$ is a zero forcing set of $Cay(G,\Omega)$ with minimum cardinality such that $B^{(0)} = B$, $B^{(1)} = \{a,a^2\}$ and $G = B^{(0)} \cup B^{(1)}$. Hence, $Pt(Cay(G,\Omega),B) = 1$ and so $Pt(Cay(G,\Omega)) = 1$.
Let $o(a) = 6$ and $B = G \setminus \{a,a^3,a^5\}$ be the set of initial black vertices of $Cay(G,\Omega)$. Then 1 forces a^3, a^4 forces a and a^2 forces a^5 in one stage. By Theorem 2.1, $Z(Cay(G,\Omega)) = n-3$. Thus B is a zero forcing set of $Cay(G,\Omega)$ with minimum cardinality. Also we have $B^{(0)} = B$, $B^{(1)} = \{a,a^3,a^5\}$

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and $G = B^{(0)} \cup B^{(1)}$. Thus $Pt(Cay(G,\Omega), B) = 1$ and so $Pt(Cay(G,\Omega)) = 1$.
Let $o(a) = 2$ and $ab = ba$. Since $b \notin \Omega$ and $\Omega = \Omega^{-1}$, $o(b) = 2$. Also let Z be a zero forcing set of $Cay(G,\Omega)$ with $Z(Cay(G,\Omega)) = |Z|$. Since $Cay(G,\Omega)$ is a vertex transitive graph, we may assume that $1 \in Z$ is a first forcing vertex. Thus there is $C \subseteq \Omega \cap Z$ such that $|C| = n-4$. Hence, $n-3 \leqslant |Z|$. Also we have $N(a) = N(b) = G \setminus \{1, ab\}$. Thus $a \in Z$ or $b \in Z$. So $n-2 \leqslant |Z|$. Since $Cay(G,\Omega)$ is not a complete graph, $Z(Cay(G,\Omega)) = n-2$. It is easy to check that $B = G \setminus \{b, ab\}$ is a zero forcing set of $Cay(G,\Omega)$ such that $B^{(0)} = B$, $B^{(1)} = \{b, ab\}$ and $G = B^{(0)} \cup B^{(1)}$. Hence $Pt(Cay(G,\Omega), B) = 1$ and so $Pt(Cay(G,\Omega)) = 1$.
Let $o(a) = 2$, $ab \neq ba$ and $o(ab) = 3$. Since $b \notin \Omega$ and $\Omega = \Omega^{-1}$, $o(b) = 2$. We have $N(a) = G \setminus \{1, ba\}$, $N(b) = G \setminus \{1, ab\}$, $aba = bab$ and $N(aba) = G \setminus \{ab, ba\}$. Let $B = G \setminus \{aba, a, b\}$ be the set of initial black vertices. In the first stage aba, a and b are forced by 1, ab and ba, respectively. By Theorem 2.1, B is a zero forcing set of $Cay(G,\Omega)$ with minimum cardinality such that $B^{(0)} = B$, $B^{(1)} = \{aba, a, b\}$ and $G = B^{(0)} \cup B^{(1)}$. Thus $Pt(Cay(G,\Omega), B) = 1$ and so $Pt(Cay(G,\Omega)) = 1$. This completes the proof.

□

Theorem 3.3. *Let $G =< \Omega >$ be a group of order $2t$, where $1 \notin \Omega = \Omega^{-1}$ and $|\Omega| = t$. If the induced subgraph on Ω in $Cay(G,\Omega)$ is isomorphic to $\overline{K_t}$, then $Pt(Cay(G,\Omega)) = 1$.*

Proof. Since $Cay(G,\Omega)$ is $|\Omega|-$regular graph, $|\Omega| = t$ and induced subgraph on Ω in $Cay(G,\Omega)$ is isomorphic to $\overline{K_t}$, $N(x) = G \setminus \Omega$ and $N(y) = \Omega$ for every $x \in \Omega$ and $y \in G \setminus \Omega$. Thus $Cay(G,\Omega)$ is isomorphic to $K_{t,t}$. Therefore, $Pt(Cay(G,\Omega) = 1$ by Lemma, 2.7. □

Theorem 3.4. *Let $G = D_{2n} = \langle a, b \mid a^n = b^2 = (ab)^2 = 1 \rangle$ be the dihedral group of order $2n$, where $n = 2k$. Also let $\Omega = \{a, a^3, \cdots, a^{2k-1}, b\}$. Then $Pt(Cay(D_{2n},\Omega)) = 1$.*

Proof. By the proof of Theorem 2.9 in [11], $Cay(D_{2n},\Omega)$ is a matching graph. Since $Z(Cay(D_{2n},\Omega)) = n$, $Pt(Cay(D_{2n},\Omega)) = 1$ by Theorem 2.5. □

Theorem 3.5. *Let $G = \langle a \rangle$ be a cyclic group of order $2n$. If n is odd and $\Omega = \{a^{2k} \mid 1 \leqslant k \leqslant n-1\} \cup \{a^n\}$, then $Z(Cay(G,\Omega)) = n$ and $Pt(Cay(G,\Omega)) = 1$.*

Proof. Let $V_1 = \{a^{2k+1} \mid 0 \leqslant k \leqslant n-1\}$ and $V_2 = \{a^{2k} \mid 0 \leqslant k \leqslant n-1\}$. Then the induced subgraphs on V_1 and V_2 are isomorphic to K_n and $Cay(G,\Omega)$ is isomorphic to a graph having the structure given in Figure 1. So $Cay(G,\Omega)$ is a matching graph.

Now let V_2 be the set of initial black vertices of $Cay(G,\Omega)$. Then for every $0 \leqslant k \leqslant n-1$, a^{2k} forces a^{2k+n}. Thus $Z(Cay(G,\Omega)) \leqslant |V_2| = n$. By Theorem 2.1, $Z(Cay(G,\Omega)) = n$. Since $Cay(G,\Omega)$ is a matching graph and $|G| = 2Z(Cay(G,\Omega))$, by Theorem 2.5, $Pt(Cay(G,\Omega)) = 1$.

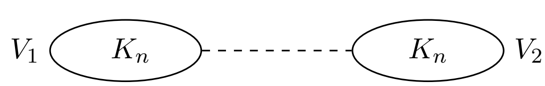

FIGURE 1
Dashed line: Every vertex of V_1 is adjacent to exactly one vertex of V_2.

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□

Theorem 3.6. *Let $G = \langle a \rangle$ be a cyclic group of order $2n$ and $\Omega = \{a^{2k+1} \mid 0 \leqslant k \leqslant n-1\}$. Then $Pt(Cay(G,\Omega)) = 1$.*

Proof. It is easy to see that $G \backslash \Omega$ is a subgroup of G. By Lemma 2.8, $Cay(G,\Omega)$ is a complete bipartite graph. By Lemma 2.7, $Pt(Cay(G,\Omega)) = 1$. □

Theorem 3.7. *Let $G = \langle a \rangle$ be a cyclic group of order $2n$, where n is even. If $\Omega = \{a^{2k+1} \mid 0 \leqslant k \leqslant n-1\} \cup \{a^n\}$, then $Z(Cay(G,\Omega)) = \frac{3n}{2}$ and $Pt(Cay(G,\Omega)) = 1$.*

Proof. For every $k \in \{0, 1, \cdots, n-1\}$, we have
$N(a^{2k+1}) = \{a^{2k+1+n}\} \cup \{a^{2j} \mid 1 \leqslant j \leqslant n\}$, and $N(a^{2k}) = \{a^{2k+n}\} \cup \Omega \setminus \{a^n\}$.
If $V_1 = \{a^{2k+1} \mid 0 \leqslant k \leqslant \frac{n}{2} - 1\}$, $V_2 = \{a^{2k} \mid 0 \leqslant k \leqslant \frac{n}{2} - 1\}$, $V_3 = \{a^{2k+1} \mid \frac{n}{2} \leqslant k \leqslant n-1\}$ and $V_4 = \{a^{2k} \mid \frac{n}{2} \leqslant k \leqslant n-1\}$, then the induced subgraph on V_i is isomorphic to $\overline{K_{\frac{n}{2}}}$ for $1 \leqslant i \leqslant 4$ and $Cay(G,\Omega)$ is isomorphic to a graph having the structure given in Figure 2.

Let Z be a zero forcing set of $Cay(G,\Omega)$ such that $Z(Cay(G,\Omega)) = |Z|$. We may assume that $1 \in Z$ is a first forcing vertex. Thus there is $C \subseteq \Omega \cap Z$ such that $|C| = n$. Without loss of generality, let $C = \Omega \setminus \{a^n\}$. Then $n+1 \leqslant |Z|$. Now if $|Z \cap \{a^{2k} \mid 1 \leqslant k \leqslant n-1, k \neq \frac{n}{2}\}| < \frac{n-2}{2}$, then there is $1 \leqslant j \leqslant n-1$ such that $a^{2j} \in V_2$ and $a^{2j+n} \in V_4$ are not in Z. Since a^{2j} is adjacent to a^{2j+n}, they are not forced by any vertices. Which is a contradiction. Hence,

$$|Z \cap \{a^{2k} \mid 1 \leqslant k \leqslant n-1, k \neq \frac{n}{2}\}| \geqslant \frac{n-2}{2}.$$

So $\frac{3n}{2} = n + 1 + \frac{n-2}{2} \leqslant |Z|$. Now let $B = V_1 \cup V_2 \cup V_3$ be the set of initial black vertices in $Cay(G,\Omega)$. Then 1 forces a^n. Since for every $1 \leqslant k \leqslant \frac{n-2}{2}$, a^{n+2k} is the only white adjacent vertex a^{2k}, a^{2k} forces a^{n+2k}. Thus B is a zero forcing set of $Cay(G,\Omega)$ and so $Z(Cay(G,\Omega)) \leqslant \frac{3n}{2}$. Thus $Z(Cay(G,\Omega)) = \frac{3n}{2}$. Also we have $B^{(0)} = B$, $B^{(1)} = V_4$ and $G = B^{(0)} \cup B^{(1)}$. Hence, $Pt(Cay(G,\Omega), B) = 1$. Therefore $Pt(Cay(G,\Omega)) = 1$.

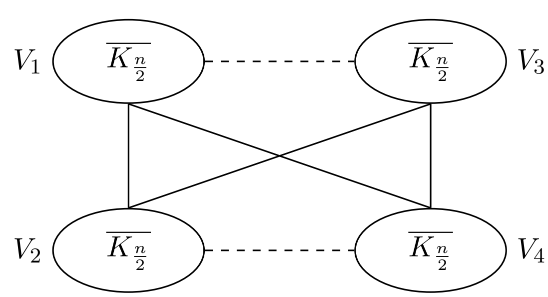

FIGURE 2

Bold line: Every vertex of the set is adjacent to every vertex of the other set.

Dashed line: Every vertex of the set is adjacent to exactly one vertex of the other set.

□

Theorem 3.8. *Let $G = \langle a \rangle$ be a cyclic group of order $2n$, where n is odd and $\Omega = \{a^{2k+1} \mid 0 \leqslant k \leqslant n-1\} \setminus \{a^n\}$. Then $Z(Cay(C_n,\Omega)) = 2n - 4$ and $Pt(Cay(G,\Omega)) = 1$.*

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Proof. Let $V_1 = \{a^{2k+1} \mid 0 \leqslant k \leqslant n-1\} \setminus \{a^n\}$ and $V_2 = \{a^{2k} \mid 1 \leqslant k \leqslant n-1\}$. Then the induced subgraph on V_i is isomorphic to $\overline{K_n}$, for $i \in \{1,2\}$ and $Cay(G,\Omega)$ is isomorphic to a graph having the structure given in Figure 3.

Let Z be a zero forcing set of $Cay(G,\Omega)$ with minimum cardinality. Since $Cay(G,\Omega)$ is a vertex transitive graph, we may assume that $1 \in Z$ is a first forcing vertex. Then there exists $C \subseteq V_1 \cap Z$ such that $|C| = n-2$. Thus $n-1 \leqslant |Z|$. If $|Z \cap V_2| \leqslant n-4$, then every black vertex in V_1 and a^n have at least two white neighbor vertices in V_2. This contradicts the fact that Z is a zero forcing set of $Cay(G,\Omega)$. Thus $|Z \cap V_2| \geqslant n-3$. Hence $|Z| \geqslant (n-1)+(n-3) = 2n-4$.

Now let $B = G \setminus \{a^{2n-1}, a^2, a^{n+1}, a^n\}$ be the set of initial black vertices in $Cay(G,\Omega)$. In the first stage, the vertices a^{2n-1}, a^2, a^{n+1} and a^n are forced by 1, a, a^{n+2} and a^{n-1}, respectively. Therefore, $Z(Cay(G,\Omega)) = 2n-4$. Also we have $B^{(0)} = B$, $B^{(1)} = \{a^{2n-1}, a^2, a^{n+1}, a^n\}$ and $G = B^{(0)} \cup B^{(1)}$. Therefore, $Pt(Cay(G,\Omega), B) = 1$ and so $Pt(Cay(G,\Omega)) = 1$.

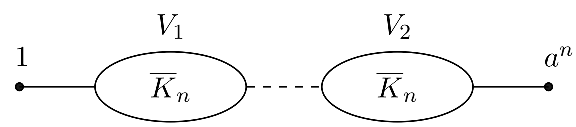

FIGURE 3

Bold line: Every vertex of the set is adjacent to every vertex of the other set.

Dashed line: Every vertex of the set is adjacent to all vertices of other set except one vertex.

□

Theorem 3.9. *Let $T_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1}\rangle$ where, n is odd and $\Omega = \{a^k \mid 1 \leqslant k \leqslant 2n-1, k \neq n\} \cup \{ab, a^{n+1}b\}$. If $n = 3$, then $Z(Cay(T_{12},\Omega)) = M(Cay(T_{12},\Omega)) = 8$. Otherwise, $Z(Cay(T_{4n},\Omega)) = M(Cay(T_{4n},\Omega)) = 3n$ and $Pt(Cay(T_{4n},\Omega)) = 1$.*

Proof. Let $n = 3$. Then by the proof of Theorem 2.10 [1], zero is an eigenvalue of $Cay(T_{12},\Omega)$ with multiplicity of 8. By Lemma 2.4, $M(Cay(T_{4n},\Omega)) \geqslant 8$. Hence $Z(Cay(T_{12},\Omega)) = M(Cay(T_{12},\Omega)) = 8$, by Lemma 2.11 and Theorem 2.3.

Let $n > 3$. Then $\Omega a^n = \Omega a^{-n} = \Omega$, because $a^n = a^{-n}$. Now let $0 \leqslant k \leqslant n-1$ and $x \in N(a^k)$. Then $xa^{-k} \in \Omega$. So $xa^{-k}a^{-n} \in \Omega a^{-n} = \Omega$. Hence, $x \in N(a^{n+k})$ and so $N(a^k) \subseteq N(a^{n+k})$. If $x \in N(a^{n+k})$, then $xa^{-n-k} \in \Omega$. Thus $xa^{-k} \in \Omega a^n = \Omega$. Hence, $x \in N(a^k)$. This shows that $N(a^{n+k}) \subseteq N(a^k)$. Therefore, $N(a^k) = N(a^{n+k})$, for $0 \leqslant k \leqslant n-1$.

By similar argument, we have $N(a^kb) = N(a^{n+k}b)$, where $0 \leqslant k \leqslant n-1$. It is easy to see that $N(a^kb) = \{a^{n-k+1}, a^{2n-k+1}\} \cup \{a^ib \mid 0 \leqslant i \leqslant 2n-1\} \setminus \{a^kb, a^{n+k}b\}$ and $N(a^k) = \{a^{n-k+1}b, a^{2n-k+1}b\} \cup < a > \setminus \{a^k, a^{n+k}\}$. Let L be a $n \times n$ matrix such that $L_{12} = L_{21} = L_{j\,(n-j+3)} = 1$, for $3 \leqslant j \leqslant n$ and the other entries are zero. It follows that :

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$$L=\begin{pmatrix}0&1&0&0&\dots&0&0\\1&0&0&0&\dots&0&0\\0&0&0&0&\dots&0&1\\0&0&0&0&\dots&1&0\\\vdots&\vdots&\vdots&\vdots&\dots&\vdots&\vdots\\0&0&0&1&\dots&0&0\\0&0&1&0&\dots&0&0\end{pmatrix}$$

Then the adjacency matrix of $Cay(T_{4n},\Omega)$, A, is as following, where if $1\leqslant i\leqslant 2n$, then $R_i(A)$ is the corresponding row of vertex $a^{i-1}$and for $2n+1\leqslant i\leqslant 4n$, $R_i(A)$ is the corresponding row of vertex $a^{i-2n-1}b$.

$$A=\left(\begin{array}{c|c|c|c}J_n-I_n&J_n-I_n&L&L\\\hline J_n-I_n&J_n-I_n&L&L\\\hline L&L&J_n-I_n&J_n-I_n\\\hline L&L&J_n-I_n&J_n-I_n\end{array}\right)$$

Now let C be a $4n\times 4n$ matrix obtained by scaling some entries of A. as following :

$$C=\left(\begin{array}{c|c|c|c}(n-1)(J_n-I_n)&(n-1)(J_n-I_n)&(n-1)L&(n-1)L\\\hline (n-1)(J_n-I_n)&(n-1)(J_n-I_n)&(n-1)L&(n-1)L\\\hline (n-1)L&(n-1)L&J_n-(n-1)I_n&J_n-(n-1)I_n\\\hline (n-1)L&(n-1)L&J_n-(n-1)I_n&J_n-(n-1)I_n\end{array}\right)$$

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Then $C \in S(Cay(T_{4n}, \Omega))$. It is not hard to see that in C, we have $(\sum_{j=0}^{n-1} R_{2n+j+1}(C)) - R_{2n+2}(C) = R_1(C)$, $\sum_{j=1}^{n-1} R_{2n+j+1}(C) = R_2(C)$ and for $2 \leqslant i \leqslant n-1$, $(\sum_{j=0}^{n-1} R_{2n+j+1}(C)) - R_{3n-i+2}(C) = R_{i+1}(C)$.
Also $N(a^k) = N(a^{n+k})$ and $N(a^k b) = N(a^{n+k} b)$ for $0 \leqslant k \leqslant n-1$ implies that $R_{k+1}(C) = R_{n+k+1}(C)$ and $R_{2n+k+1}(C) = R_{3n+k+1}(C)$. By elementary row operation, we have $null(C) \geqslant 3n$. Thus $M(Cay(T_{4n}, \Omega)) \geqslant 3n$. By Lemma 2.11 and Theorem 2.3, $M(Cay(T_{4n}, \Omega)) = Z(Cay(T_{4n}, \Omega)) = 3n$.
Now let $B = < a > \cup \{a^i b \mid 0 \leqslant i \leqslant n-1\}$ be the set of initial black vertices of $Cay(T_{4n}, \Omega)$. We saw in the proof of Lemma 2.11, B is a zero forcing set of $Cay(T_{4n}, \Omega)$ such that $Z(Cay(T_{4n}, \Omega)) = |B|$. Furthermore, $a^{n+1}b$, $a^n b$ and $a^{2n-k+1}b$ are forced by 1, a and a^k, respectively in one stage, where $2 \leqslant k \leqslant n-1$. Thus $B^{(0)} = B$ and $B^{(1)} = \{a^{n+k}b \mid 0 \leqslant k \leqslant n-1\}$. Hence, $T_{4n} = B^{(0)} \cup B^{(1)}$ and so $Pt(Cay(T_{4n}, \Omega), B) = 1$. Therefore, $Pt(Cay(T_{4n}, \Omega)) = 1$. □

Theorem 3.10. *Let $T_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1}\rangle$ where n is even and $\Omega = \{a^{2k+1} \mid 0 \leqslant k \leqslant n-1\} \cup \{b, b^{-1}\}$. Then $Z(Cay(T_{4n}, \Omega)) = 3n$ and $Pt(Cay(T_{4n}, \Omega)) = 1$.*

Proof. Let $V_1 = \{a^{2k+1} \mid 0 \leqslant k \leqslant n-1\}$, $V_2 = \{a^{2k} \mid 0 \leqslant k \leqslant n-1\}$. Then $T_{4n} = V_1 \cup V_2 \cup V_1 b \cup V_2 b$ and the induced subgraphs on V_i and $V_i b$ are isomorphic to $\overline{K_n}$, for $i \in \{1, 2\}$. If $0 \leqslant k \leqslant n-1$, then $N(a^{2k}) = V_1 \cup \{a^{2n-2k}b, a^{n-2k}b\}$, $N(a^{2k}b) = V_1 b \cup \{a^{2n-2k}, a^{n-2k}\}$, $N(a^{2k+1}) = V_2 \cup \{a^{2n-2k-1}b, a^{n-2k-1}b\}$ and $N(a^{2k+1}b) = V_2 b \cup \{a^{2n-2k-1}, a^{n-2k-1}\}$. Furthermore $Cay(T_{4n}, \Omega)$ is isomorphic to a graph having the structure given in Figure 4.

Let $X = V_1 \cup V_2 \cup \{a^{2k}b \mid 0 \leqslant k \leqslant \frac{n}{2} - 1\} \cup \{a^{2k+1}b \mid 0 \leqslant k \leqslant \frac{n}{2} - 1\}$ be the set of initial black vertices of $Cay(T_{4n}, \Omega)$. Then for every $1 \leqslant k \leqslant \frac{n}{2} - 1$, $a^{2n-2k}b$ is the only white neighbor of a^{2k} and $a^{2n-2k-1}b$ is the only white neighbor of a^{2k+1}. Thus a^{2k} forces a^{2n-2k} and a^{2k+1} forces $a^{2n-2k-1}$. Also it is clear that b^{-1} and $a^{2n-1}b$ are forced by 1 and a, respectively. Hence, X is a zero forcing set of $Cay(T_{4n}, \Omega)$. Therefore, $Z(Cay((T_{4n}, \Omega)) \leqslant 3n$.
Now let Z be a zero forcing set of $Cay(T_{4n}, \Omega)$ with cardinality at most $3n - 1$. Since $Cay(T_{4n}, \Omega)$ is a vertex transitive graph, we may assume that $1 \in Z$ is a first forcing vertex. Thus there is $C \subseteq (V_1 \cup \{b, b^{-1}\}) \cap Z$ such that $|C| = n + 1$. Without loss of generality, let $C = V_1 \cup \{b\}$. Let t_i and t'_i be the number of white vertices in V_i and $V_i b$, respectively for $i \in \{1, 2\}$. Then $t_2 + t'_1 + t'_2 = n + 1$. If $t'_1 > \frac{n}{2}$, then there is $0 \leqslant k \leqslant \frac{n}{2} - 1$ such that $a^{2k+1}b$ and $a^{n+2k+1}b$ are white vertices. Since $N(a^{2k+1}b) \cap V_1 = N(a^{n+2k+1}b) \cap V_1 = \{a^{2n-2k-1}, a^{n-2k-1}\}$, a^{n+2k+1} and $a^{2k+1}b$ are not forced by any vertices of V_1. Also every vertex in $V_2 b$ has at least two white vertices $a^{2k+1}b$ and $a^{n+2k+1}b$. Thus $a^{n+2k+1}b$ and $a^{2k+1}b$ are not forced by any vertices, which is a contradiction. Hence, $t'_1 \leqslant \frac{n}{2}$. The same argument shows $t_2 \leqslant \frac{n}{2}$ and $t'_2 \leqslant \frac{n}{2}$. If $t_2 = t'_1 = 2$, then every vertex in $V_1 \cup V_2 b$ has at least two white neighbor vertices and so the zero forcing process is stopped. Hence, $(t_2, t'_1, t'_2) \in \{(1, \frac{n}{2}, \frac{n}{2}), (\frac{n}{2}, 1, \frac{n}{2})\}$. Let $(t_2, t'_1, t'_2) = (1, \frac{n}{2}, \frac{n}{2})$ and a^{2j} be the only white vertex in V_2 for some $1 \leqslant j \leqslant n-1$. Since $V_1 b \subset N(V_2 b)$ and $t'_2 = \frac{n}{2}$, a^{2j} is forced by a vertex in V_1, which we denote a^{2i+1}. Thus, $a^{2n-2i-1}b$ and $a^{n-2i-1}b$ are black vertices. Also all of vertices in $V_1 b$ are forced by V_1. Since $N(a^{2k+1}) \cap V_1 b =$

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$\{a^{2n-2k-1}b, a^{n-2k-1}b\}$, for every $0 \leqslant k \leqslant n-1$, $a^{2n-2k-1}b \in Z$ or $a^{n-2k-1}b \in Z$. We have $t'_1 = \frac{n}{2}$, so for every $0 \leqslant k \leqslant n-1$ if $a^{2n-2k-1} \in Z$, then $a^{n-2k-1} \notin Z$ (or if $a^{n-2k-1} \in Z$, then $a^{2n-2k-1} \notin Z$). This is contradiction by this fact that $a^{2n-2i-1} \in Z$ and $a^{n-2i-1} \in Z$.
Let $(t_2, t'_1, t'_2) = (\frac{n}{2}, 1, \frac{n}{2})$. The same argument runs as before. Therefore, $Z(Cay$ $(T_{4n}, \Omega)) = 3n$.
Let $B = V_1 \cup V_2 \cup \{a^{2k}b, a^{2k+1}b \mid 0 \leqslant k \leqslant \frac{n}{2}-1\}$ be the set of initial black vertices in $Cay(T_{4n}, \Omega)$. In one stage the vertices of V_1 force $\{a^{2k+1}b \mid \frac{n}{2} \leqslant k \leqslant n-1\}$ and the vertices of V_2 force $\{a^{2k}b \mid \frac{n}{2} \leqslant k \leqslant n-1\}$. Thus $B^{(1)} = \{a^{2k}b, a^{2k+1}b \mid \frac{n}{2} \leqslant k \leqslant n-1\}$. Hence $T_{4n} = B^{(0)} \cup B^{(1)}$ and so $Pt(Cay(T_{4n}, \Omega), B) = 1$. Therefore, $Pt(Cay(T_{4n}, \Omega)) = 1$.

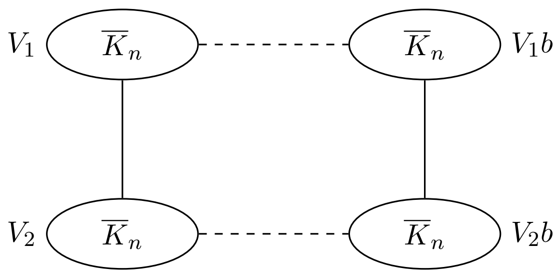

FIGURE 4

Bold line: Every vertex of the set is adjacent to every vertices of the other set.

Dashed line: Every vertex of the set is adjacent to exactly two vertices of the other set.

□

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