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ON A HILBERT-TYPE INTEGRAL INEQUALITY IN THE WHOLE PLANE

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Abstract. Using weight functions and techniques of real analysis, a new Hilbert-type integral inequality in the whole plane with nonhomogeneous kernel and a best possible constant factor is proved. Equivalent forms, several particular inequalities and operator expressions are considered.

1. Introduction

If $f(x), g(y) \geq 0$,

$$0 < \int_0^\infty f^2(x)dx < \infty \quad \text{and} \quad 0 < \int_0^\infty g^2(y)dy < \infty,$$

then the following well known Hilbert integral inequality (cf. [1]) is satisfied:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi \left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(y)dy \right)^{\frac{1}{2}}, \tag{1.1}$$

where the constant factor π is the best possible.

Recently, by the use of weight functions and introducing multi-parameters, several extensions of (1.1) were presented in Yang's books (cf. [2], [3]). Some Hilbert-type inequalities with homogenous kernels of degree 0 and non-homogenous kernels were established in [4]- [7]. Some other kinds of Hilbert-type inequalities were obtained in [8]- [18]. Many of these inequalities are constructed in the quarter plane of the first quadrant.

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Using weight functions, Yang in 2007 [19] proved the following Hilbert-type integral inequality in the whole plane:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)}{(1+e^{x+y})^{\lambda}}dxdy$$
$$< \quad B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left(\int_{-\infty}^{\infty}e^{-\lambda x}f^{2}(x)dx\int_{-\infty}^{\infty}e^{-\lambda y}g^{2}(y)dy\right)^{\frac{1}{2}}, \tag{1.2}$$

where the constant factor $B(\frac{\lambda}{2},\frac{\lambda}{2})(\lambda>0)$ is the best possible, and $B(u,v)$ stands for the beta function (cf. [20]). He et al. [21]- [36] proved some new Hilbert-type inequalities in the whole plane with the best possible constant factors. The methods in these papers are demanding and interesting.

In the present paper, still by means of weight functions, by introducing a nonhomogeneous kernel, a new Hilbert-type integral inequality in the whole plane with multi-parameters and a best possible constant factor is proved. Equivalent forms, some particular inequalities and operator expressions are also considered. The lemmas and theorems provide an extensive account of this type of inequalities.

2. Some lemmas

In the sequel, we assume that

$$0<\alpha_1\leq\alpha_2<\pi,\mu,\sigma>0,\mu+\sigma=\lambda,\delta\in\{-1,1\},\gamma\in\{a;a=\frac{1}{2k+1},2k-1\,(k\in\mathbf{N}=\{1,2,\cdots\})\}.$$

Definition 2.1. We define two weight functions as follows:
For $x,y\in\mathbf{R}=(-\infty,\infty)$,

$$\omega(\sigma,y):=\int_{-\infty}^{\infty}\max_{i\in\{1,2\}}\frac{|y|^{\sigma}|x|^{\delta\sigma-1}}{[\max\{|x^{\delta}y|^{\gamma}+(x^{\delta}y)^{\gamma}\cos\alpha_i,1\}]^{\lambda/\gamma}}dx, \tag{2.1}$$

$$\varpi(\sigma,x):=\int_{-\infty}^{\infty}\max_{i\in\{1,2\}}\frac{|x|^{\delta\sigma}|y|^{\sigma-1}}{[\max\{|x^{\delta}y|^{\gamma}+(x^{\delta}y)^{\gamma}\cos\alpha_i,1\}]^{\lambda/\gamma}}dy. \tag{2.2}$$

Lemma 2.2. *We have the following expressions:*

$$\omega(\sigma,y)=\varpi(\sigma,x)=K(\sigma)\ (y,x\in\mathbf{R}\backslash\{0\}), \tag{2.3}$$

$$K(\sigma):=\frac{1}{2^{\frac{\sigma}{\gamma}}}[(\sec\frac{\alpha_2}{2})^{\frac{2\sigma}{\gamma}}+(\csc\frac{\alpha_1}{2})^{\frac{2\sigma}{\gamma}}]\frac{\lambda}{\mu\sigma}\in\mathbf{R}_{+}=(0,\infty). \tag{2.4}$$

Proof. For $y\in\mathbf{R}\backslash\{0\}$, setting $u=x^{\delta}y$ in (2.1), we derive that

$$x=y^{\frac{-1}{\delta}}u^{\frac{1}{\delta}},\ dx=\frac{1}{\delta}y^{\frac{-1}{\delta}}u^{\frac{1}{\delta}-1}du$$

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and

$$
\begin{aligned}
\omega(\sigma, y) &= |\frac{1}{\delta}| \int_{-\infty}^{\infty} \max_{i\in\{1,2\}} \frac{1}{(\max\{|u|^{\gamma} + u^{\gamma}\cos\alpha_i, 1\})^{\lambda/\gamma}} |u|^{\sigma-1} du \\
&= K_1(\sigma) + K_2(\sigma), \qquad (2.5)\\
K_1(\sigma) &:= \int_{-\infty}^{0} \max_{i\in\{1,2\}} \frac{1}{[\max\{(-u)^{\gamma}(1-\cos\alpha_i), 1\}]^{\lambda/\gamma}} (-u)^{\sigma-1} du, \\
K_2(\sigma) &:= \int_{0}^{\infty} \max_{i\in\{1,2\}} \frac{1}{[\max\{u^{\gamma}(1+\cos\alpha_i), 1\}]^{\lambda/\gamma}} u^{\sigma-1} du.
\end{aligned}
$$

Setting $v = u^{\gamma}(1+\cos\alpha_i)$ in the integral of $K_2(\sigma)$, we obtain that

$$
u = \frac{1}{(1+\cos\alpha_i)^{1/\gamma}} v^{\frac{1}{\gamma}}, \quad du = \frac{1}{\gamma(1+\cos\alpha_i)^{1/\gamma}} v^{\frac{1}{\gamma}-1} dv
$$

and

$$
\begin{aligned}
K_2(\sigma) &= \int_{0}^{\infty} \max_{i\in\{1,2\}} \frac{1}{\gamma(1+\cos\alpha_i)^{\sigma/\gamma}} \frac{1}{(\max\{v, 1\})^{\lambda/\gamma}} v^{\frac{\sigma}{\gamma}-1} dv \\
&= \frac{1}{\gamma(1+\cos\alpha_2)^{\sigma/\gamma}} \int_{0}^{\infty} \frac{1}{(\max\{v, 1\})^{\lambda/\gamma}} v^{\frac{\sigma}{\gamma}-1} dv \\
&= \frac{1}{\gamma(1+\cos\alpha_2)^{\sigma/\gamma}} \left(\int_{0}^{1} v^{\frac{\sigma}{\gamma}-1} dv + \int_{1}^{\infty} \frac{1}{v^{\lambda/\gamma}} v^{\frac{\sigma}{\gamma}-1} dv \right) \\
&= \frac{1}{(1+\cos\alpha_2)^{\sigma/\gamma}} \frac{\lambda}{\mu\sigma} = \frac{1}{2^{\sigma/\gamma}} (\sec\frac{\alpha_2}{2})^{\frac{2\sigma}{\gamma}} \frac{\lambda}{\mu\sigma} \in \mathbf{R}_+.
\end{aligned}
$$

Setting $v = -u$ in the integral of $K_1(\sigma)$, we similarly obtain that

$$
\begin{aligned}
K_1(\sigma) &= \int_{0}^{\infty} \max_{i\in\{1,2\}} \frac{1}{\{\max\{v^{\gamma}[1+\cos(\pi-\alpha_i)], 1\}\}^{\lambda/\gamma}} v^{\sigma-1} dv \\
&= \frac{1}{[1+\cos(\pi-\alpha_1)]^{\sigma/\gamma}} \frac{\lambda}{\mu\sigma} = \frac{1}{2^{\sigma/\gamma}} (\csc\frac{\alpha_1}{2})^{\frac{2\sigma}{\gamma}} \frac{\lambda}{\mu\sigma} \in \mathbf{R}_+,
\end{aligned}
$$

namely, we have $\omega(\sigma, y) = K(\sigma) \in \mathbf{R}_+$.

In the same way, for $x \in \mathbf{R}\backslash\{0\}$, setting $u = x^{\delta} y$ in (2.2), we get

$$
y = x^{-\delta} u, \quad dy = x^{-\delta} du
$$

and

$$
\varpi(\sigma, x) = \int_{-\infty}^{\infty} \max_{i\in\{1,2\}} \frac{1}{(\max\{|u|^{\gamma} + u^{\gamma}\cos\alpha_i, 1\})^{\lambda/\gamma}} |u|^{\sigma-1} du = K(\sigma).
$$

Hence, (2.3) follows.

This completes the proof of the lemma. □

Remark 2.3. If we replace $\max_{i\in\{1,2\}}$ by $\min_{i\in\{1,2\}}$ in (2.1) and (2.2), then (2.4) is valid by exchanging α_1 and α_2.

Lemma 2.4. *Suppose that $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $K(\sigma)$ is defined by (2.4), $f(x)$ is a non-negative measurable function in $\mathbf{R}$. We have the following inequality:*

$$\begin{aligned} J &:= \int_{-\infty}^{\infty}|y|^{p\sigma-1}\left\{\int_{-\infty}^{\infty}\max_{i\in\{1,2\}}\frac{f(x)}{[\max\{|x^{\delta}y|^{\gamma}+(x^{\delta}y)^{\gamma}\cos\alpha_i,1\}]^{\frac{\lambda}{\gamma}}}dx\right\}^{p}dy \\ &\leq K^{p}(\sigma)\int_{-\infty}^{\infty}|x|^{p(1-\delta\sigma)-1}f^{p}(x)dx. \end{aligned} \tag{2.6}$$

Proof. In the following, for simplicity, we set

$$H^{(\delta)}(x,y):=\max_{i\in\{1,2\}}\frac{1}{[\max\{|x^{\delta}y|^{\gamma}+(x^{\delta}y)^{\gamma}\cos\alpha_i,1\}]^{\lambda/\gamma}}\ (x,y\in\mathbf{R}). \tag{2.7}$$

By Hölder's inequality (cf. [37]), we obtain that

$$\begin{aligned} &\left(\int_{-\infty}^{\infty}H^{(\delta)}(x,y)f(x)dx\right)^{p} \\ &= \left\{\int_{-\infty}^{\infty}H^{(\delta)}(x,y)[\frac{|x|^{(1-\delta\sigma)/q}}{|y|^{(1-\sigma)/p}}f(x)][\frac{|y|^{(1-\sigma)/p}}{|x|^{(1-\delta\sigma)/q}}]dx\right\}^{p} \\ &\leq \int_{-\infty}^{\infty}H^{(\delta)}(x,y)\frac{|x|^{(1-\delta\sigma)(p-1)}}{|y|^{1-\sigma}}f^{p}(x)dx \\ &\quad\times\left[\int_{-\infty}^{\infty}H^{(\delta)}(x,y)\frac{|y|^{(1-\sigma)(q-1)}}{|x|^{1-\delta\sigma}}dx\right]^{p-1} \quad (2.8) \\ &= \frac{(\omega(\sigma,y))^{p-1}}{|y|^{p\sigma-1}}\int_{-\infty}^{\infty}H^{(\delta)}(x,y)\frac{|x|^{(1-\delta\sigma)(p-1)}}{|y|^{1-\sigma}}f^{p}(x)dx. \end{aligned}$$

Then by (2.3) and Fubini's theorem (cf. [38]), we have

$$\begin{aligned} J &\leq K^{p-1}(\sigma)\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}H^{(\delta)}(x,y)\frac{|x|^{(1-\delta\sigma)(p-1)}}{|y|^{1-\sigma}}f^{p}(x)dx\right]dy \\ &= K^{p-1}(\sigma)\int_{-\infty}^{\infty}\varpi(\sigma,x)|x|^{p(1-\delta\sigma)-1}f^{p}(x)dx. \end{aligned}$$

Still by (2.3), we obtain inequality (2.6).

This completes the proof of the lemma. □

3. Main results and some corollaries

Theorem 3.1. *If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $K(\sigma)$ is defined by (2.4), $f(x),g(y)\geq 0$,*

$$0<\int_{-\infty}^{\infty}|x|^{p(1-\delta\sigma)-1}f^{p}(x)dx<\infty \ \ and \ \ 0<\int_{-\infty}^{\infty}|y|^{q(1-\sigma)-1}g^{q}(y)dy<\infty,$$

then we have the following equivalent inequalities:

$$I:=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\max_{i\in\{1,2\}}\frac{f(x)g(y)}{(\max\{|x^{\delta}y|^{\gamma}+(x^{\delta}y)^{\gamma}\cos\alpha_i,1\})^{\frac{\lambda}{\gamma}}}dxdy$$

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$$< K(\sigma)\left[\int_{-\infty}^{\infty}|x|^{p(1-\delta\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}|y|^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}}, \tag{3.1}$$

$$J \ := \int_{-\infty}^{\infty}|y|^{p\sigma-1}\left\{\int_{-\infty}^{\infty}\max_{i\in\{1,2\}}\frac{f(x)}{[\max\{|x^\delta y|^\gamma+(x^\delta y)^\gamma\cos\alpha_i,1\}]^{\frac{\lambda}{\gamma}}}dx\right\}^p dy$$

$$< K^p(\sigma)\int_{-\infty}^{\infty}|x|^{p(1-\delta\sigma)-1}f^p(x)dx, \tag{3.2}$$

where the constant factors $K(\sigma)$ and $K^p(\sigma)$ are the best possible.

In particular, for $\alpha_1=\alpha_2=\alpha\in(0,\pi)$, $\gamma=1$ in (3.1) and (3.2), setting

$$k(\sigma):=\frac{1}{2^\sigma}[(\sec\frac{\alpha}{2})^{2\sigma}+(\csc\frac{\alpha}{2})^{2\sigma}]\frac{\lambda}{\mu\sigma}, \tag{3.3}$$

we deduce (3.3) and (3.4) to the following equivalent inequalities:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(\max\{|x^\delta y|+x^\delta y\cos\alpha,1\})^\lambda}f(x)g(y)dxdy$$

$$< k(\sigma)\left[\int_{-\infty}^{\infty}|x|^{p(1-\delta\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}|y|^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}}, \tag{3.4}$$

$$\int_{-\infty}^{\infty}|y|^{p\sigma-1}\left[\int_{-\infty}^{\infty}\frac{1}{(\max\{|x^\delta y|+x^\delta y\cos\alpha,1\})^\lambda}f(x)dx\right]^p dy$$

$$< k^p(\sigma)\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1}f^p(x)dx. \tag{3.5}$$

Proof. If (2.8) takes the form of equality for a $y\neq 0$, then there exist constants A and B, satisfying $A^2+B^2>0$, and

$$A\frac{|x|^{(1-\delta\sigma)(p-1)}}{|y|^{1-\sigma}}f^p(x)=B\frac{|y|^{(1-\sigma)(q-1)}}{|x|^{1-\delta\sigma}} \text{ a. e. in } \mathbf{R}$$

(cf. [37]). We have $A\neq 0$ (otherwise, $B=A=0$), and it follows that

$$|x|^{p(1-\delta\sigma)-1}f^p(x)=|y|^{q(1-\sigma)}\frac{B}{A|x|} \text{ a.e. in } \mathbf{R},$$

which contradicts the fact that

$$0<\int_{-\infty}^{\infty}|x|^{p(1-\delta\sigma)-1}f^p(x)dx<\infty.$$

Hence, (2.8) takes the form of strict inequality. So does (2.6), namely, (3.2) follows.

In view of Hölder's inequality (cf. [37]), we also obtain that

$$I \ = \int_{-\infty}^{\infty}\left(|y|^{\sigma-\frac{1}{p}}\int_{-\infty}^{\infty}H^{(\delta)}(x,y)f(x)dx\right)(|y|^{\frac{1}{p}-\sigma}g(y))dy$$

$$\leq J^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}|y|^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}}. \tag{3.6}$$

Then by (3.2), we derive (3.1). On the other hand, assuming that (3.1) is valid, we set

$$g(y):=|y|^{p\sigma-1}\left(\int_{-\infty}^{\infty}H^{(\delta)}(x,y)f(x)dx\right)^{p-1} \ (y\in\mathbf{R}).$$

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Then it follows that

$$J = \int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y) dy.$$

In view of (2.7), we have $J < \infty$. If $J = 0$, then (3.2) is trivially valid; if $0 < J < \infty$, then in view of (3.1), we derive that

$$\begin{aligned} &\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y) dy = J = I \\ &< K(\sigma) \left[\int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned} \tag{3.7}$$

$$J^{\frac{1}{p}} = \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{p}} < K(\sigma) \left[\int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}, \tag{3.8}$$

namely, (3.2) follows, which is equivalent to (3.1).

We set $E_\delta := \{x \in \mathbf{R}; |x|^\delta \geq 1\}$, and

$$E_\delta^+ := E_\delta \cap \mathbf{R}_+ = \{x \in \mathbf{R}_+; x^\delta \geq 1\}.$$

For any $\varepsilon > 0$, we define $\tilde{f}(x), \tilde{g}(y)$ as follows:

$$\tilde{f}(x) \quad : \quad = \begin{cases} |x|^{\delta(\sigma - \frac{2\varepsilon}{p})-1}, & x \in E_\delta \\ 0, & x \in \mathbf{R} \backslash E_\delta \end{cases},$$

$$\tilde{g}(y) \quad : \quad = \begin{cases} 0, & y \in (-\infty, -1) \cup (1, \infty) \\ |y|^{\sigma + \frac{2\varepsilon}{q} - 1}, & y \in [-1, 1] \end{cases}.$$

The we obtain that

$$\begin{aligned} \tilde{L} \quad : \quad &= \left[\int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} \tilde{f}^p(x) dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} \tilde{g}^q(y) dy \right]^{\frac{1}{q}} \\ &= 2 \left(\int_{E_\delta^+} x^{-2\delta\varepsilon - 1} dx \right)^{\frac{1}{p}} \left(\int_0^1 y^{2\varepsilon - 1} dy \right)^{\frac{1}{q}} = \frac{1}{\varepsilon}. \end{aligned}$$

We have

$$\begin{aligned} h(x) : \quad &= \int_{-1}^{1} \max_{i \in \{1,2\}} \frac{|y|^{\sigma + \frac{2\varepsilon}{q} - 1}}{[\max\{|x^\delta y|^\gamma + (x^\delta y)^\gamma \cos\alpha_i, 1\}]^{\lambda/\gamma}} dy \\ &= \int_{-1}^{1} \max_{i \in \{1,2\}} \frac{|Y|^{\sigma + \frac{2\varepsilon}{q} - 1}}{[\max\{|-x^\delta Y|^\gamma + (-x^\delta Y)^\gamma \cos\alpha_i, 1\}]^{\lambda/\gamma}} dY \\ &= h(-x), \end{aligned}$$

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namely $h(x)$ is an even function. Since

$$\begin{aligned}\tilde{I} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H^{(\delta)}(x,y)\tilde{f}(x)\tilde{g}(y)dxdy \\ &= \int_{E_\delta} |x|^{\delta(\sigma-\frac{2\varepsilon}{p})-1} h(x)dx = 2\int_{E_\delta^+} x^{\delta(\sigma-\frac{2\varepsilon}{p})-1} h(x)dx \\ &= 2\int_{E_\delta^+} x^{-2\delta\varepsilon-1}\left[\int_{-x^\delta}^{x^\delta} \max_{i\in\{1,2\}} \frac{|u|^{\sigma+\frac{2\varepsilon}{q}-1}}{(\max\{|u|^\gamma + u^\gamma\cos\alpha_i, 1\})^{\lambda/\gamma}} du\right] dx,\end{aligned}$$

setting $v = x^\delta$ in the above integral, by Fubini's theorem (cf. [38]), we obtain that

$$\begin{aligned}\tilde{I} &= 2\int_1^\infty v^{-2\varepsilon-1}\left[\int_{-v}^{v} \max_{i\in\{1,2\}} \frac{|u|^{\sigma+\frac{2\varepsilon}{q}-1}}{(\max\{|u|^\gamma + u^\gamma\cos\alpha_i, 1\})^{\lambda/\gamma}} du\right] dv \\ &= 2\int_1^\infty v^{-2\varepsilon-1}\left\{\int_0^v [\max_{i\in\{1,2\}} \frac{1}{(\max\{u^\gamma(1+\cos\alpha_i), 1\})^{\lambda/\gamma}}\right. \\ &\quad \left. + \max_{i\in\{1,2\}} \frac{1}{(\max\{u^\gamma(1-\cos\alpha_i), 1\})^{\lambda/\gamma}}] u^{\sigma+\frac{2\varepsilon}{q}-1} du\right\} dv \\ &= 2\int_1^\infty v^{-2\varepsilon-1}\left\{\int_0^v [\frac{1}{(\max\{u^\gamma(1+\cos\alpha_2), 1\})^{\lambda/\gamma}}\right. \\ &\quad \left. + \frac{1}{(\max\{u^\gamma(1-\cos\alpha_1), 1\})^{\lambda/\gamma}}] u^{\sigma+\frac{2\varepsilon}{q}-1} du\right\} dv \\ &= 2\int_1^\infty v^{-2\varepsilon-1}\left\{\int_0^1 [\frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{(\max\{u^\gamma(1+\cos\alpha_2), 1\})^{\lambda/\gamma}}\right. \\ &\quad \left. + \frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{(\max\{u^\gamma(1-\cos\alpha_1), 1\})^{\lambda/\gamma}}] du\right\} dv \\ &\quad + 2\int_1^\infty v^{-2\varepsilon-1}\left\{\int_1^v [\frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{(\max\{u^\gamma(1+\cos\alpha_2), 1\})^{\lambda/\gamma}}\right. \\ &\quad \left. + \frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{(\max\{u^\gamma(1-\cos\alpha_1), 1\})^{\lambda/\gamma}}] du\right\} dv\end{aligned}$$

$$\begin{aligned}&= \frac{1}{\varepsilon}\int_0^1 \left\{\frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[\max\{u^\gamma(1+\cos\alpha_2), 1\}]^{\frac{\lambda}{\gamma}}}\right. \\ &\quad \left. + \frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[\max\{u^\gamma(1-\cos\alpha_1), 1\}]^{\frac{\lambda}{\gamma}}}\right\} du + 2\int_1^\infty \left(\int_u^\infty v^{-2\varepsilon-1} dv\right) \\ &\quad \times \left\{\frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[\max\{u^\gamma(1+\cos\alpha_2), 1\}]^{\lambda/\gamma}} + \frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[\max\{u^\gamma(1-\cos\alpha_1), 1\}]^{\lambda/\gamma}}\right\} du\end{aligned}$$

$$= \frac{1}{\varepsilon}\left\{\int_0^1 \left[\frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{(\max\{u(1+\cos\alpha_2)^{1/\gamma}, 1\})^\lambda} + \frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{(\max\{u(1-\cos\alpha_1)^{1/\gamma}, 1\})^\lambda}\right] du\right.$$

$$+\int_1^\infty \left[\frac{u^{\sigma-\frac{2\varepsilon}{p}-1}}{(\max\{u(1+\cos\alpha_2)^{1/\gamma},1\})^\lambda}+\frac{u^{\sigma-\frac{2\varepsilon}{p}-1}}{(\max\{u(1-\cos\alpha_1)^{1/\gamma},1\})^\lambda}\right]du\Bigg\}.$$

If the constant factor $K(\sigma)$ in (3.1) is not the best possible, then there exists a positive constant $k \le K(\sigma)$, such that (3.1) is valid, when replacing $K(\sigma)$ by k. In particular, we have $\varepsilon\tilde{I} < \varepsilon k\tilde{L}$, and

$$\begin{aligned}
&\int_0^1 \left\{\frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[\max\{u(1+\cos\alpha_2)^{1/\gamma},1\}]^\lambda}+\frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[\max\{u(1-\cos\alpha_1)^{1/\gamma},1\}]^\lambda}\right\}du\\
&+\int_1^\infty \left\{\frac{u^{\sigma-\frac{2\varepsilon}{p}-1}}{[\max\{u(1+\cos\alpha_2)^{1/\gamma},1\}]^\lambda}+\frac{u^{\sigma-\frac{2\varepsilon}{p}-1}}{[\max\{u(1-\cos\alpha_1)^{1/\gamma},1\}]^\lambda}\right\}du\\
&= \varepsilon\tilde{I} < \varepsilon k\tilde{L} = k. \qquad (3.9)
\end{aligned}$$

By (2.5) and Levi's theorem (cf. [38]), we deduce that

$$\begin{aligned}
K(\sigma) &= K_2(\sigma)+K_1(\sigma) = \int_0^\infty \max_{i\in\{1,2\}} \frac{1}{[\max\{u^\gamma(1+\cos\alpha_i),1\}]^{\lambda/\gamma}}u^{\sigma-1}du\\
&\quad+\int_0^\infty \max_{i\in\{1,2\}} \frac{1}{[\max\{u^\gamma(1-\cos\alpha_i),1\}]^{\lambda/\gamma}}u^{\sigma-1}du\\
&= \int_0^\infty \frac{u^{\sigma-1}du}{[\max\{u(1+\cos\alpha_2)^{\frac{1}{\gamma}},1\}]^\lambda}+\int_0^\infty \frac{u^{\sigma-1}du}{[\max\{u(1-\cos\alpha_1)^{\frac{1}{\gamma}},1\}]^\lambda}\\
&= \int_0^1 \lim_{\varepsilon\to0^+}\left\{\frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[\max\{u(1+\cos\alpha_2)^{1/\gamma},1\}]^\lambda}+\frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{[\max\{u(1-\cos\alpha_1)^{1/\gamma},1\}]^\lambda}\right\}du\\
&+\int_1^\infty \lim_{\varepsilon\to0^+}\left\{\frac{u^{\sigma-\frac{2\varepsilon}{p}-1}}{[\max\{u(1+\cos\alpha_2)^{1/\gamma},1\}]^\lambda}+\frac{u^{\sigma-\frac{2\varepsilon}{p}-1}}{[\max\{u(1-\cos\alpha_1)^{1/\gamma},1\}]^\lambda}\right\}du\\
&= \lim_{\varepsilon\to0^+}\Bigg\{\int_0^1 \left[\frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{(\max\{u(1+\cos\alpha_2)^{1/\gamma},1\})^\lambda}+\frac{u^{\sigma+\frac{2\varepsilon}{q}-1}}{(\max\{u(1-\cos\alpha_1)^{1/\gamma},1\})^\lambda}\right]du\\
&\quad+\int_1^\infty \left[\frac{u^{\sigma-\frac{2\varepsilon}{p}-1}}{(\max\{u(1+\cos\alpha_2)^{1/\gamma},1\})^\lambda}+\frac{u^{\sigma-\frac{2\varepsilon}{p}-1}}{(\max\{u(1-\cos\alpha_1)^{1/\gamma},1\})^\lambda}\right]du\Bigg\}\\
&\le k.
\end{aligned}$$

Hence, the constant factor $k = K(\sigma)$ in (3.1) is the best possible.

The constant factor in (3.2) is still the best possible. Otherwise, we would reach a contradiction by (3.6), that the constant factor in (3.1) is not the best possible.

This completes the proof of the theorem. □

Corollary 3.2. *For $\delta = 1$ in (3.1) and (3.2), we obtain the following equivalent inequalities with non-homogeneous kernel:*

$$\begin{aligned}
&\int_{-\infty}^\infty\int_{-\infty}^\infty \max_{i\in\{1,2\}} \frac{f(x)g(y)}{(\max\{|xy|^\gamma+(xy)^\gamma\cos\alpha_i,1\})^{\frac{\lambda}{\gamma}}}dxdy\\
&< K(\sigma)\left[\int_{-\infty}^\infty |x|^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^\infty |y|^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}}, \qquad (3.10)
\end{aligned}$$

$$\int_{-\infty}^{\infty} |y|^{p\sigma-1} \left\{ \int_{-\infty}^{\infty} \max_{i\in\{1,2\}} \frac{f(x)}{[\max\{|xy|^{\gamma}+(xy)^{\gamma}\cos\alpha_i, 1\}]^{\frac{\lambda}{\gamma}}} dx \right\}^p dy$$

$$(3.11) \qquad < K^p(\sigma) \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx,$$

where the constant factors $K(\sigma)$ *and* $K^p(\sigma)$ *are the best possible. In particular, for* $\alpha_1 = \alpha_2 = \alpha \in (0, \pi)$, $\gamma = 1$ *in* (3.10) *and* (3.11), *we have the following equivalent inequalities:*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x)g(y)}{(\max\{|xy|+xy\cos\alpha, 1\})^{\lambda}} dx dy$$

$$(3.12) \qquad < k(\sigma) \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}},$$

$$\int_{-\infty}^{\infty} |y|^{p\sigma-1} \left[\int_{-\infty}^{\infty} \frac{f(x)}{(\max\{|xy|+xy\cos\alpha, 1\})^{\lambda}} dx \right]^p dy$$

$$(3.13) \qquad < k^p(\sigma) \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx,$$

where $k(\sigma)$ *is defined by* (3.3).

Corollary 3.3. *For* $\delta = -1$ *in* (3.1) *and* (3.2), *replacing* $|x|^{\lambda} f(x)$ *by* $f(x)$, *we obtain*

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x) dx < \infty,$$

and the following equivalent inequalities with homogeneous kernel:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \max_{i\in\{1,2\}} \frac{f(x)g(y)}{(\max\{|y|^{\gamma}+sgn(x)y^{\gamma}\cos\alpha_i, |x|^{\gamma}\})^{\lambda/\gamma}} dx dy$$

$$(3.14) \qquad < K(\sigma) \left[\int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}},$$

$$\int_{-\infty}^{\infty} |y|^{p\sigma-1} \left[\int_{-\infty}^{\infty} \max_{i\in\{1,2\}} \frac{f(x)}{(\max\{|y|^{\gamma}+sgn(x)y^{\gamma}\cos\alpha_i, |x|^{\gamma}\})^{\lambda/\gamma}} dx \right]^p dy$$

$$(3.15) \qquad < K^p(\sigma) \int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x) dx,$$

where the constant factors $K(\sigma)$ *and* $K^p(\sigma)$ *are the best possible. In particular, for* $\alpha_1 = \alpha_2 = \alpha \in (0, \pi)$, $\gamma = 1$ *in* (3.14) *and* (3.15), *we obtain the following equivalent inequalities:*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(\max\{|y|+sgn(x)y\cos\alpha, |x|\})^{\lambda}} f(x)g(y) dx dy$$

$$(3.16) \qquad < k(\sigma) \left[\int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}},$$

$$\int_{-\infty}^{\infty} |y|^{p\sigma-1} \left[\int_{-\infty}^{\infty} \frac{1}{(\max\{|y|+sgn(x)y\cos\alpha, |x|\})^{\lambda}} f(x) dx \right]^p dy$$

$$(3.17) \qquad < k^p(\sigma) \int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x) dx,$$

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where $k(\sigma)$ is defined by (3.3).

4. **Operator expressions**

Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1$. We set the following functions:

$$\varphi(x) := |x|^{p(1-\delta\sigma)-1}, \psi(y) := |y|^{q(1-\sigma)-1}, \phi(x) := |x|^{p(1-\mu)-1} (x, y \in \mathbf{R}),$$

wherefrom, $\psi^{1-p}(y) = |y|^{p\sigma-1}$. Define the following real normed linear space:

$$\begin{aligned} L_{p,\varphi}(\mathbf{R}) \quad : &= \left\{ f : ||f||_{p,\varphi} := \left(\int_{-\infty}^{\infty} \varphi(x)|f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\}, \\ L_{p,\psi^{1-p}}(\mathbf{R}) \quad &= \quad \left\{ h : ||h||_{p,\psi^{1-p}} = \left(\int_{-\infty}^{\infty} \psi^{1-p}(y)|h(y)|^p dy \right)^{\frac{1}{p}} < \infty \right\}, \\ L_{p,\phi}(\mathbf{R}) \quad &= \quad \left\{ g : ||g||_{p,\phi} = \left(\int_{-\infty}^{\infty} \phi(x)|g(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\}. \end{aligned}$$

(a) In view of Theorem 1, for $f \in L_{p,\varphi}(\mathbf{R})$, setting

$$H_1(y) := \int_{-\infty}^{\infty} \max_{i \in \{1,2\}} \frac{|f(x)|}{[\max\{|x^\delta y|^\gamma + (x^\delta y)^\gamma \cos\alpha_i, 1\}]^{\frac{\lambda}{\gamma}}} dx \quad (y \in \mathbf{R}),$$

by (3.2), we have

$$||H_1||_{p,\psi^{1-p}} := \left(\int_{-\infty}^{\infty} \psi^{1-p}(y) H_1^p(y) dy \right)^{\frac{1}{p}} < K(\sigma)||f||_{p,\varphi} < \infty. \tag{4.1}$$

Definition 4.1. We define a Hilbert-type integral operator $T_1 : L_{p,\varphi}(\mathbf{R}) \to L_{p,\psi^{1-p}}(\mathbf{R})$ with non-homogeneous kernel in the whole plane as follows: For any $f \in L_{p,\varphi}(\mathbf{R})$, there exists a unique representation $T_1 f = H_1 \in L_{p,\psi^{1-p}}(\mathbf{R})$, satisfying $T_1 f(y) = H_1(y)$, for any $y \in \mathbf{R}$.

In view of (4.1), it follows that

$$||T_1 f||_{p,\psi^{1-p}} = ||H_1||_{p,\psi^{1-p}} \leq K(\sigma)||f||_{p,\varphi},$$

and then the operator T_1 is bounded satisfying

$$||T_1|| = \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R})} \frac{||T_1 f||_{p,\psi^{1-p}}}{||f||_{p,\varphi}} \leq K(\sigma).$$

Since the constant factor $K(\sigma)$ in (4.1) is the best possible, we have $||T_1|| = K(\sigma)$.

If we define the formal inner product of $T_1 f$ and g as follows:

$$\begin{aligned} (T_1 f, g) \quad : &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} H^{(\delta)}(x,y) f(x) dx \right) g(y) dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H^{(\delta)}(x,y) f(x) g(y) dx dy, \end{aligned}$$

then we can rewrite (3.1) and (3.2) as follows:

$$(T_1 f, g) < ||T_1|| \cdot ||f||_{p,\varphi} ||g||_{q,\psi}, ||T_1 f||_{p,\psi^{1-p}} < ||T_1|| \cdot ||f||_{p,\varphi}.$$

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(b) In view of Corollary 2, for $f \in L_{p,\phi}(\mathbf{R})$, setting

$$H_2(y) := \int_{-\infty}^{\infty} \max_{i\in\{1,2\}} \frac{|f(x)|}{(\max\{|y|^\gamma + sgn(x)y^\gamma \cos\alpha_i, |x|^\gamma\})^{\lambda/\gamma}} dx (y \in \mathbf{R}),$$

by (3.15), we have

$$||H_2||_{p,\psi^{1-p}} := \left(\int_{-\infty}^{\infty} \psi^{1-p}(y) H_2^p(y) dy\right)^{\frac{1}{p}} < K(\sigma)||f||_{p,\phi} < \infty. \tag{4.2}$$

Definition 4.2. Define a Hilbert-type integral operator $T_2 : L_{p,\phi}(\mathbf{R}) \to L_{p,\psi^{1-p}}(\mathbf{R})$ with homogeneous kernel in the whole plane as follows: For any $f \in L_{p,\phi}(\mathbf{R})$, there exists a unique representation $T_2 f = H_2 \in L_{p,\psi^{1-p}}(\mathbf{R})$, satisfying $T_2 f(y) = H_2(y)$, for any $y \in \mathbf{R}$.

In view of (4.2), it follows that

$$||T_2 f||_{p,\psi^{1-p}} = ||H_2||_{p,\psi^{1-p}} \le K(\sigma)||f||_{p,\phi},$$

and then the operator T_2 is bounded satisfying

$$||T_2|| = \sup_{f(\neq\theta)\in L_{p,\phi}(\mathbf{R})} \frac{||T_2 f||_{p,\psi^{1-p}}}{||f||_{p,\phi}} \le K(\sigma).$$

Since the constant factor $K(\sigma)$ in (4.2) is the best possible, we have $||T_2|| = K(\sigma)$.

If we define the formal inner product of $T_2 f$ and g as

$$(T_2 f, g) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \max_{i\in\{1,2\}} \frac{f(x)g(y)}{(\max\{|y|^\gamma + sgn(x)y^\gamma \cos\alpha_i, |x|^\gamma\})^{\frac{\lambda}{\gamma}}} dxdy,$$

then we can rewrite (3.14) and (3.15) as follows:

$$(T_2 f, g) < ||T_2|| \cdot ||f||_{p,\phi}||g||_{q,\psi}, ||T_2 f||_{p,\psi^{1-p}} < ||T_2|| \cdot ||f||_{p,\phi}.$$

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References

[1] G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities, Cambridge University Press, Cambridge, USA, 1934.

[2] B.C. Yang, The Norm of Operator and Hilbert-Type Inequalities, Science Press, Beijing, China, 2009.

[3] B.C. Yang, Hilbert-Type Integral Inequalities, Bentham Science Publishers Ltd., The United Arab Emirates, 2009

[4] B.C. Yang, On the norm of an integral operator and applications, *J. Math. Anal. Appl.* **321** (2006), no. 1, 182–192.

[5] B.C. Yang, On the norm of a Hilbert's type linear operator and applications, *J. Math. Anal. Appl.* **325** (2007), no. 1, 529–541.

[6] B.C. Yang, A Hilbert-type integral inequality with a homogeneous kernel of degree zero, (Chinese) *J. Shandong Univ. Nat. Sci.* **45** (2010), no. 2, 103–106.

[7] L. Debnath, B.C. Yang, Recent developments of Hilbert-type discrete and integral inequalities with applications, *Int. J. Math. Math. Sci.* 2012, Art. ID 871845, 29 pages.

[8] M. Th. Rassias, B.C. Yang, On half-discrete Hilbert's inequality, *Appl. Math. Comput.* **220** (2013) 75–93.

DOI: https://10.30504/JIMS.2021.309063.1044

[9] B.C. Yang, M. Krnić, A half-discrete Hilbert-type inequality with a general homogeneous kernel of degree 0, *J. Math. Inequal.* **6**(2012), no. 3, 401–417.

[10] M. Krnić, P. Vuković, Multidimensional Hilbert-type inequalities obtained via local fractional calculus, *Acta Appl. Math.* **169** (2020), 667–680. doi:10.1007/s10440-020-00317-x

[11] V. Adiyasuren, T. Batbold, and M. Krnić, Half-discrete Hilbert-type inequalities with mean operators, the best constants, and applications, Applied Mathematics and Computation, **231** (2014), 148–159 doi:10.1016/j.amc.2014.01.011

[12] V. Adiyasuren, T. Batbold, and M. Krnić, On several new Hilbert-type inequalities involving means operators, *Acta Math. Sin. (Engl. Ser.)* **29** (2013), no. 8, 1493–1514.

[13] I. Brnetić, M. Krnić, and J. Pečarić, Multiple Hilbert and Hardy-Hilbert inequalities with non-conjugate parameters, *Bull. Austral. Math. Soc.* **71** (2005), no. 3, 447–457.

[14] Th.M. Rassias, B.C. Yang, A multidimensional half-discrete Hilbert-type inequality and the Riemann zeta function, *Appl. Math. Comput.* **225** (2013), 263–277.

[15] M.Th. Rassias, B.C. Yang, On a multidimensional half-discrete Hilbert-type inequality related to the hyperbolic cotangent function, *Appl. Math. Comput.* **242** (2014), 800–813.

[16] C.J. Zhao, W.S. Cheung, On Hilbert's inequalities with alternating signs, *J. Math. Inequal* **12** (2018), no. 1, 191–200.

[17] P. Gao, On weighted Hardy inequalities for non-increasing sequences, *J. Math. Inequal.* **12** (2018), no. 2, 551–557.

[18] M. Th. Rassias and B.C. Yang, On an equivalent property of a reverse Hilbert-type integral inequality related to the extended Hurwitz-zeta function, *J. Math. Inequal.* **13** (2019), no. 2, 315–334.

[19] B.C. Yang, A new Hilbert-type integral inequality, Soochow *J. Math.* **33** (2007), no. 4, 849–859.

[20] Z.Q. Wang, D.R. Guo, Introduction to special functions, Science Press, Beijing, China, 1979.

[21] B. He, B.C. Yang, On a Hilbert-type integral inequality with the homogeneous kernel of 0-degree and the hypergeometrc function, *Mathematics in Practice and Theory* **40** (2010), no. 18, 105–211.

[22] B.C. Yang, A new Hilbert-type integral inequality with some parameters, *J. Jilin Univ. (Science Edition)* **46** (2008), no. 6, 1085–1090.

[23] B.C. Yang, A Hilbert-type integral inequality with a nonhomogeneous kernel, (Chinese) *Xiamen Daxue Xuebao Ziran Kexue Ban* **48** (2009), no. 2, 165–169.

[24] B.C. Yang, A reverse Hilbert-type integral inequality with some parameters, *J. Xinxiang Univ. (Natural Science Edition),* **27** (2010), no. 6, 1–4.

[25] D.M. Xin, B.C. Yang, A Hilbert-type integral inequality in the whole plane with the homogeneous kernel of degree -2, *J. Inequal. Appl.* 2011, Art. ID 401428, 11 pages.

[26] B. Yang, A reverse Hilbert-type integral inequality with a nonhomogeneous kernel, (Chinese) *J. Jilin Univ. Sci.* **49** (2011), no. 3, 437–441.

[27] Z.T. Xie, Z. Zeng, Y.F. Sun, A new Hilbert-type inequality with the homogeneous kernel of degree-2, *Adv. Appl. Math. Sci.* **12** (2013), no. 7, 391–401.

[28] Z. Zhen, K. Raja Rama Gandhi, Z.T. Xie, A new Hilbert-type inequality with the homogeneous kernel of degree -2 and with the integral, *Bull. Math. Sci. App.* **3** (2014), no. 1, 11–20.

[29] M.Th. Rassias, B.C. Yang,A Hilbert-type integral inequality in the whole plane related to the hypergeometric function and the beta function, *J. Math. Anal. Appl.* **428** (2015), no. 2, 1286–1308.

[30] Z.H. Gu, B.C. Yang, A Hilbert-type integral inequality in the whole plane with a non-homogeneous kernel and a few parameters, *J. Inequal. Appl.* (2015), 2015:314, 9 pages.

[31] M.Th. Rassias, B.C. Yang, On a Hilbert-type integral inequality in the whole plane related to the extended Riemann zeta function, *Complex Anal. Oper. Theory* **13** (2019), no. 4, 1765–1782. https://doi.org/10.1007/s11785-018-0830-5.

[32] M.Th. Rassias, B.C. Yang, A. Raigorodskii,On Hardy-type integral inequalities in the whole plane related to the extended Hurwitz-zeta function, *J. Inequal. Appl.* (2020), Paper No. 94, 24 pages.

[33] M.Th. Rassias, B.C. Yang, A. Raigorodskii, On the reverse Hardy-type integral inequalities in the whole plane with the extended Riemann-zeta function, *J. Math. Inequal.* **14** (2020), no. 2, 525–546.

[34] M.Th. Rassias, B.C. Yang, A. Raigorodskii, On a more accurate reverse Hilbert-type inequality in the whole plane, *J. Math. Inequal.* **14** (2020), no. 4, 1359–1374.

[35] M.Th. Rassias, B.C. Yang, A. Raigorodskii, A Hilbert-type integral inequality in the whole plane related to the Arc tangent function, Symmetry 2021, 13, 351. https://doi.org/10.3390/sym13020351.

[36] M.Th. Rassias, B.C. Yang, G.C.Meletiou, A more accurate half-discrete Hilbert-type inequality in the whole plane and the reverses, *Ann. Funct. Anal.* (2021), no. 3, Paper No. 50, 29 pages.

[37] J.C. Kuang, Applied inequalities, Shangdong Science and Technology Press, Jinan, China, 2004.

[38] J.C. Kuang, Introudction to Real Analysis, Hunan Educiton Press, Changsha, China, 1996.

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