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SHING-TUNG YAU'S WORK ON THE NOTION OF MASS IN GENERAL RELATIVITY

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The notion of mass or energy has been one of the most challenging problems in general relativity since Einstein's time. As is well known from the equivalence principle, there is no well-defined concept of energy density for gravitation. On the other hand, when there is asymptotic symmetry, concepts of total energy and momentum can be defined. This is the ADM energy-momentum and the Bondi energy-momentum when the system is viewed from spatial infinity and null infinity, respectively. These concepts are fundamental in general relativity but there are limitations to such definitions if the physical system is not isolated and cannot quite be viewed from infinity where asymptotic symmetry exists.

The positive energy conjecture states that the total energy of a nontrivial isolated physical system must be positive. This conjecture lies in the foundation of general relativity upon which stability of the system rests. This long standing conjecture had attracted many physicists and mathematician, but only very special cases were verified up until the seventies. It was finally resolved by Schoen and Yau in the affirmative [1]. The positive mass theorem, as it is often called in the mathematical community, was immediately applied to the study of conformal geometry and nonlinear partial differential equations. In particular, it was the key ingredient in Schoen's solution of the celebrated Yamabe problem. There is rarely a mathematical theorem in the modern time that has such profound influences on both physics and mathematics. In the following, we discuss Yau's representative work that are related to the notion of mass in general relativity.

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(1) Positive mass theorem

In [2], Schoen and Yau first settled the major case of the positive mass theorem for spacetimes with a maximal slice. The idea of the proof goes back to earlier works of Schoen-Yau on the study of three-manifolds of non-negative scalar curvature. Schoen and Yau dealt with the general conjecture in [3]. They combined an equation proposed by Jang and the conformal method to reduce the proof to the previous case. Jang observed that the equation is not solvable in general and a solution may blow up. Schoen and Yau discovered that the blow-up loci of the solution are exactly the locations of apparent horizons on the initial data set. Schoen-Yau's solution of the Jang's equation became an important tool in the study of existence of black holes and quasi-local mass. Schoen and Yau also extended their method to prove the positivity of the Bondi mass [4], which is the total mass of an isolated physical system measured after the loss due to the gravitational radiation. Later Witten gave a different proof of the positive energy theorem using ideas from quantum gravity. Subsequently, various types of positive mass theorem are proved generalizing the methods of Schoen-Yau and Witten.

(2) Definition of the center of mass.

In [6], Huisken and Yau constructed constant mean curvature surface foliation on the exterior region of an asymptotically flat end with strictly positive mass. The results allowed them to define a geometric center of mass and provided a natural coordinates system near infinity in an intrinsic way.

(3) Existence of black holes.

By the singularity theorem of Penrose, a closed trapped surface on the initial data set will evolve into a black hole singularity. In [5], Schoen and Yau established that on a bounded region of an initial data set, if the mass density is large enough on a large region, an apparent horizon (and thus a closed trapped surface) is destined to form inside the region. In [7], Yau showed if the boundary effect (in terms of the boundary mean curvature) is strong enough, even if the matter density is negative, black hole can still form. This result is closely related to Yau's later work on quasi-local mass.

(4) Quasi-local mass

In 1982, Penrose proposed a list of major unsolved problems in general relativity, and the first was "find a suitable quasilocal definition of energy-momentum (mass)". This concept is associated with a spacelike 2-surface in spacetime and it measures the total energy contained in the enclosed region. There have been many attempts to define such a notion in the past decades. There are several empirical criterion such a notion has to satisfy. It obviously has to have the right asymptotical behavior at spatial and null infinity to recover the ADM and Bondi mass, respectively. The most subtle and difficult requirements are that the quasi-local mass should be positive in general and zero for surfaces in the Minkowski space. In the joint work of Wang and Yau in [8,9], they finally find a definition of quasilocal mass that satisfies all the requirements necessary for a valid definition. They explore the Hamilton-Jacobi analysis of the gravitational action by Brown and York and use isometric embeddings into Minkowski space to anchor the reference surface Hamiltonian.

To sum up, Yau's work not only resolve fundamental problems in general relativity and discover new physical phenomena by geometric means, but also provide fresh insight and perspective into classical concepts. His vision, idea, and techniques revolutionized the field and continue to have positive impact on its future development.

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