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CANONICAL SECTIONS OF HODGE BUNDLES ON MODULI SPACES

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Abstract. We review recent works in [K. Liu, S. Rao and X. Yang, Quasi-isometry and deformations of Calabi–Yau manifolds, *Invent. Math.* 199 (2015), no. 2, 423–453.] and [K. Liu and Y. Shen, Moduli spaces as ball quotients I, local theory, Preprint] on geometry of sections of Hodge bundles and their applications to moduli spaces.

1. Introduction

This paper presents a review of some recent works on deformation theory and Hodge theory in [19] and [23].

In some sense, deformation theory is the study of the infinitesimal structures of moduli spaces, while Hodge theory studies the global structures of these spaces. The two methods can also be combined. We can use the method in deformation theory to construct global sections of Hodge bundles as in [19], and use the method in Hodge theory to study finer structures of local expansions of sections of Hodge bundles as in [23].

The geometry of Hodge bundles, which is equivalent to variation of Hodge structure in some sense, determines the geometry of moduli spaces. In [23], the present author gave an intrinsic characterization of the polarized manifolds whose moduli space can be realized locally as a ball quotient, which generalized the work of Allcock, Carlson, and Toledo in [1] and [2]. Using methods of Hodge theory,

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the local expansions of sections of Hodge bundles for such polarized manifolds are proved to be linear functions in certain special coordinates, which implies that the corresponding period domain has a sub-domain isomorphic to a complex ball.

The paper is organized as follows. In Section 2, we review the work in [19], and describe the global sections of the Hodge bundles for Calabi–Yau manifolds in Theorem 2.3. In Section 3, we review the definitions and basic properties of period domains and period maps used in local expansions of sections of Hodge bundles. In Theorem 4.2 of Section 4, we give the sections of Hodge bundles for Calabi–Yau type manifolds, with expansions up to order two. We sketch the proof for special cases, which reiterates the main ideas in [23].

In Section 5, we apply the expansion formulas in Section 4 to certain polarized manifolds, here called the polarized manifolds of ball type. In particular, Theorem 5.2 gives the linear expansions of the sections of the Hodge bundles for polarized manifolds of ball type in certain special coordinates. Then we prove in Corollary 5.3 that the corresponding moduli space is locally isomorphic to the ball quotient. At the end of this section, we review some examples in [23] of the polarized manifolds of ball type.

2. Canonical sections of Hodge bundles for Calabi–Yau manifolds

First let us recall some notions from deformation theory due to Kodaira and Spencer. A family of compact complex manifolds is given by a proper holomorphic map

$$\pi : \mathcal{X} \to S$$

between complex manifolds $\mathcal{X}$ and S, such that for any $s \in S$ the fiber

$$X_s \triangleq \pi^{-1}(s)$$

is a compact complex manifold.

We assume that $S = \Delta$ is an open disc of the complex plane $\mathbb{C}$ containing 0. Denote the fiber X_0 by X. For $p \geq 0$, we denote by $A^{0,p}(X, \mathrm{T}^{1,0}_X)$ ($Z^{0,p}(X, \mathrm{T}^{1,0}_X)$, resp.) the space of global sections of $(0, p)$-forms (closed $(0, p)$-forms„> resp.) with values in $\mathrm{T}^{1,0}_X$, the holomorphic tangent bundle of X. Then the complex structure on X_t, for $t \in \Delta$, is determined by a smooth section,

$$\Phi(t) \in A^{0,1}(X, \mathrm{T}^{1,0}_X)$$

satisfying the following conditions:

(1) $\Phi(t)$ is holomorphic in $t \in \Delta$ with $\Phi(0) = 0$. Moreover, in the expansion

$$\Phi(t) = \phi_1 t + \phi_2 t^2 + \cdots + \phi_k t^k + \cdots$$

we have $\phi_1 \in Z^{0,1}(X, \mathrm{T}^{1,0}_X)$;

(2) $\Phi(t)$ satisfies the integrability equation

$$\bar{\partial}\Phi(t) = \frac{1}{2}[\Phi(t), \Phi(t)], \tag{2.1}$$

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which is equivalent to

$$\overline{\partial}\phi_1 = 0,$$

$$\overline{\partial}\phi_k = \frac{1}{2}\sum_{i=1}^{k-1}[\phi_i, \phi_{k-i}],\ k \geq 2. \tag{2.2}$$

The Kodaira-Spencer map ρ is then defined by

$$\mathrm{T}_0\Delta \to H^1(X, \Theta_X),\ \frac{\partial}{\partial t} \mapsto [\phi_1]$$

where $\Theta_X = \mathcal{O}_X(\mathrm{T}_X^{1,0})$, and $[\phi_1]$ is the cohomology class of ϕ_1 in the quotient space,

$$\frac{Z^{0,1}(X, \mathrm{T}_X^{1,0})}{\overline{\partial}A^{0,0}(X, \mathrm{T}_X^{1,0})},$$

which is isomorphic to $H^1(X, \Theta_X)$, the first sheaf cohomology group of the tangent sheaf. Sometimes it is convenient to use the space $\mathbb{H}^{0,1}(X, \mathrm{T}_X^{1,0})$ of harmonic $(0,1)$-forms with values in $\mathrm{T}_X^{1,0}$, which is isomorphic to $H^1(X, \Theta_X)$. In this case, we can choose $\phi_1 \in \mathbb{H}^{0,1}(X, \mathrm{T}_X^{1,0})$.

Conversely, given $\Phi(t) \in A^{0,1}(X, \mathrm{T}_X^{1,0})$ satisfying conditions (1) and (2), one can construct a family $\pi : \mathcal{X} \to S$ of compact complex manifolds. In general, the integrability condition (2.1) has obstructions in the cohomology group $H^2(X, \Theta_X)$.

Let X be a Calabi–Yau manifold of complex dimension $n \geq 3$, which means that X is a projective manifold with a trivial canonical bundle

$$\Omega_X^n \simeq \mathcal{O}_X$$

and satisfies $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < n$. Then $\dim_{\mathbb{C}} H^0(X, \Omega_X^n) = 1$ with one generator Ω, which is a nowhere vanishing holomorphic $(n, 0)$-form on X. The unobstructedness of the deformation of Calabi–Yau manifolds is proved in [27] and [28], which is equivalent to solving the $\overline{\partial}$-equations (2.2) for $|t|$ small enough.

In [19], the first author and coauthors gave a global solution to $\overline{\partial}$-equations (2.2) for Calabi–Yau manifolds, by using the quasi-isometry inequalities for compact Kähler manifolds.

Theorem 2.1 (Theorem 4.3 of [19])**.** *Let X be a Calabi–Yau manifold and $\phi_1 \in \mathbb{H}^{0,1}(X, \mathrm{T}_X^{1,0})$ with $|\phi_1|_{C^1} < C$ for some constant C. Then, there exits a smooth globally convergent power series for $|t| < 1$,*

$$\Phi(t) = \phi_1 t + \phi_2 t^2 + \cdots + \phi_k t^k + \cdots$$

such that the integrable equation (2.1) holds.

The method of the proof is to extract more information by using quasi-isometry formula, in each step of solving the $\overline{\partial}$-equations (2.2),

(1) $\overline{\partial}^* \phi_k = 0$, for $k \geq 1$;
(2) $\phi_k \lrcorner \Omega_0$ is ∂-exact for $k \geq 2$.

Here Ω_0 is a generator of $H^0(X, \Omega_X^n)$.

Corollary 2.2. *Under the conditions of Theorem 2.1, we have $||\Phi(t) \lrcorner \Omega_0||_{L^2} < \infty$, $|t| < 1$.*

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The above results remain valid in the more general case of $S = \Delta$ being a polydisc in $\mathbb{C}^m$. If we write $t = (t_1, \cdots, t_m)$ and

$$\Phi(t) = \Phi(t_1, \cdots, t_m)$$

with power series

$$\begin{aligned} \Phi(t) &= \sum_{\substack{\nu_1, \cdots, \nu_m \geq 0 \\ \nu_1 + \cdots + \nu_m \neq 0}} \varphi_{\nu_1 \cdots \nu_m} t_1^{\nu_1} \cdots t_m^{\nu_m} \\ &\triangleq \phi_1 t + \phi_2 t^2 + \cdots + \phi_k t^k + \cdots \end{aligned}$$

where

$$\phi_k t^k \triangleq \sum_{\nu_1 + \cdots + \nu_m = k} \varphi_{\nu_1 \cdots \nu_m} t_1^{\nu_1} \cdots t_m^{\nu_m}.$$

Corollary 2.2 implies that the L^2-norm of

$$e^{\Phi(t)} \lrcorner \Omega_0 := \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{\Phi(t) \lrcorner \cdots \Phi(t) \lrcorner}_{k \text{ times}} \Omega_0$$

is finite. By a direct computation, we see that

$$e^{\Phi(t)} \lrcorner : A^{n,0}(X) \to A^{n,0}(X_t) \tag{2.3}$$

is a well-defined isomorphism for $|t| < 1$. Let us put

$$\Omega^C(t) = e^{\Phi(t)} \lrcorner \Omega_0.$$

Then $\Omega^C(t) \in A^{n,0}(X_t)$. Moreover, we have the following result.

Theorem 2.3 (Theorem 5.2, Corollary 5.3 of [19])**.** *Let Ω_0 be a nontrivial holomorphic $(n,0)$-form on the Calabi–Yau manifold X. Then the global L^2 family $\Omega^C(t) = e^{\Phi(t)} \lrcorner \Omega_0$ is holomorphic in t for $|t| < 1$ such that $[\Omega^C(t)] \in H^{n,0}(X_t)$. Moreover, we have the following global expansion of $[\Omega^C(t)]$ in cohomology class,*

$$[\Omega^C(t)] = [\Omega_0] + \sum_{i=1}^{m} [\varphi_i \lrcorner \Omega_0] t_i + O(|t|^2) \tag{2.4}$$

where $O(|t|^2)$ denotes the terms in $\bigoplus_{j=2}^n H^{n-j,j}(X)$ of order ≥ 2.

Here $[\cdot]$ denotes the corresponding cohomology class.

3. Period maps and period domains

In order to apply the results in Section 2 to more general cases, we need the method of period maps in Hodge theory. In this section we review the definitions and basic properties of period domains and period maps.

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Let $H_{\mathbb{Z}}$ be a fixed lattice and $H = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ be its complexification. Let n be a positive integer, and Q be a bilinear form on $H_{\mathbb{Z}}$ which is symmetric if n is even and skew-symmetric if n is odd. Let $h^{i,n-i}$, $0 \leq i \leq n$, be integers such that

$$\sum_{i=0}^{n} h^{i,n-i} = \dim_{\mathbb{C}} H.$$

The period domain D for the polarized Hodge structures of type

$$\{H_{\mathbb{Z}}, Q, h^{i,n-i}\}$$

is the set of all collections of the subspaces $H^{i,n-i}$, $0 \leq i \leq n$, of H such that

$$H = \bigoplus_{0 \leq i \leq n} H^{i,n-i},\ H^{i,n-i} = \overline{H^{n-i,i}},\ \dim_{\mathbb{C}} H^{i,n-i} = h^{i,n-i} \text{ for } 0 \leq i \leq n,$$

and on which Q satisfies the Hodge-Riemann bilinear relations:

$$Q\left(H^{i,n-i}, H^{j,n-j}\right) = 0 \text{ unless } i + j = n; \tag{3.1}$$

$$\left(\sqrt{-1}\right)^{2k-n} Q\left(v, \overline{v}\right) > 0 \text{ for } v \in H^{k,n-k} \setminus \{0\}. \tag{3.2}$$

Alternatively, in terms of Hodge filtrations, the period domain D is the set of all collections of the filtrations

$$H = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^n,$$

such that

$$\begin{aligned} &\dim_{\mathbb{C}} F^i = f^i, \\ &H = F^i \oplus \overline{F^{n-i+1}}, \text{ for } 0 \leq i \leq n, \end{aligned} \tag{3.3}$$

where $f^i = h^{n,0} + \cdots + h^{i,n-i}$, and on which Q satisfies the Hodge-Riemann bilinear relations in the form of Hodge filtrations

$$Q\left(F^i, F^{n-i+1}\right) = 0; \tag{3.4}$$

$$Q\left(Cv, \overline{v}\right) > 0 \text{ if } v \neq 0, \tag{3.5}$$

where C is the Weil operator given by

$$Cv = \left(\sqrt{-1}\right)^{2k-n} v$$

for $v \in F^k \cap \overline{F^{n-k}}$.

Let $\Phi : S \to \Gamma \backslash D$ be a period map from geometry. More precisely we have an algebraic family

$$f : \mathcal{X} \to S$$

of polarized algebraic manifolds over a quasi-projective manifold S, such that for any $q \in S$, the point $\Phi(q)$, modulo certain action of the monodromy group Γ, represents the Hodge structure of the n-th primitive cohomology group $H^n_{\mathrm{pr}}(X_q, \mathbb{C})$ of the fiber $X_q = f^{-1}(q)$. Here $H \simeq H^n_{\mathrm{pr}}(X_q, \mathbb{C})$ for any $q \in S$.

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Since period map is locally liftable, we can lift the period map to $\Phi : \mathcal{T} \to D$ by taking the universal cover $\mathcal{T}$ of S such that the diagram

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\Phi"] \arrow[d, "\pi"] & D \arrow[d, "\pi"] \\ S \arrow[r, "\Phi"] & \Gamma\backslash D \end{tikzcd} \tag{3.6}$$

is commutative.

Let $q \in \mathcal{T}$ be any point. We denote the Hodge filtration by

$$H^n_{\mathrm{pr}}(X_q, \mathbb{C}) = F^0_q \supseteq F^1_q \supseteq \cdots \supseteq F^n_q$$

with $F^i_q = F^i H^n_{\mathrm{pr}}(X_q, \mathbb{C}) = H^{n,0}_{\mathrm{pr}}(X_q) \oplus \cdots \oplus H^{i,n-i}_{\mathrm{pr}}(X_q)$ for $0 \leq i \leq n$.

In this paper, all the vectors are taken as column vectors, since we will consider the left actions on the period domains. To simplify notations, we use row vectors and transpose them to make column vectors.

Let us introduce the notion of adapted basis for the given Hodge decomposition or Hodge filtration. We call a basis

$$\xi = \left\{\xi_0, \cdots, \xi_{f^n-1}, \xi_{f^n}, \cdots, \xi_{f^{n-1}-1}, \cdots, \xi_{f^{k+1}}, \cdots, \xi_{f^k-1}, \cdots, \xi_{f^1}, \cdots, \xi_{f^0-1}\right\}^T$$

of $H^n_{\mathrm{pr}}(X_q, \mathbb{C})$ an adapted basis for the given Hodge decomposition if it satisfies

$$H^{k,n-k}_{\mathrm{pr}}(X_q) = \mathrm{Span}_{\mathbb{C}}\left\{\xi_{f^{k+1}}, \cdots, \xi_{f^k-1}\right\}, \ 0 \leq k \leq n.$$

We call a basis

$$\zeta = \left(\zeta_0, \cdots, \zeta_{f^n-1}, \zeta_{f^n}, \cdots, \zeta_{f^{n-1}-1}, \cdots, \zeta_{f^{k+1}}, \cdots, \zeta_{f^k-1}, \cdots, \zeta_{f^1}, \cdots, \zeta_{f^0-1}\right)^T$$

of $H^n_{\mathrm{pr}}(X_q, \mathbb{C})$ an adapted basis for the given filtration if it satisfies

$$F^k_q = \mathrm{Span}_{\mathbb{C}}\{\zeta_0, \cdots, \zeta_{f^k-1}\}, \ 0 \leq k \leq n.$$

For convenience, we set $f^{n+1} = 0$ and $m = f^0$.

Remark 3.1. The adapted basis at the base point p can be chosen with respect to the given Hodge decomposition or the given Hodge filtration. While, in order that the period map is holomorphic, the adapted basis at any other point q can only be chosen with respect to the given Hodge filtration.

Definition 3.2. (1) Let

$$\xi = \left(\xi_0, \cdots, \xi_{f^n-1}, \cdots, \xi_{f^{k+1}}, \cdots, \xi_{f^k-1}, \cdots, \xi_{f^1}, \cdots, \xi_{f^0-1}\right)^T$$

be the adapted basis with respect to the Hodge decomposition or the Hodge filtration at any point. The blocks of ξ are defined by

$$\xi_{(\alpha)} = \left(\xi_{f^{-\alpha+n+1}}, \cdots, \xi_{f^{-\alpha+n}-1}\right)^T,$$

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for $0 \le \alpha \le n$. Then

$$\xi = (\xi_{(0)}^T, \cdots, \xi_{(n)}^T)^T = \begin{pmatrix} \xi_{(0)} \\ \vdots \\ \xi_{(n)} \end{pmatrix}.$$

(2) The blocks of an $m \times m$ matrix $\Psi = (\Psi_{ij})_{0 \le i,j \le m-1}$ are set as follows. For each $0 \le \alpha, \beta \le n$, the (α, β)-th block $\Psi^{(\alpha,\beta)}$ is defined by

$$\Psi^{(\alpha,\beta)} = (\Psi_{ij})_{f^{-\alpha+n+1} \le i \le f^{-\alpha+n}-1,\ f^{-\beta+n+1} \le j \le f^{-\beta+n}-1}. \tag{3.7}$$

In particular, $\Psi = (\Psi^{(\alpha,\beta)})_{0 \le \alpha,\beta \le n}$ is called a block upper (lower, resp.) triangular matrix if $\Psi^{(\alpha,\beta)} = 0$ whenever $\alpha > \beta$ ($\alpha < \beta$, resp.).

Let $H_{\mathbb{F}} = H^n_{\mathrm{pr}}(X, \mathbb{F})$, where $\mathbb{F}$ can be chosen as $\mathbb{Z}$, $\mathbb{R}$ or $\mathbb{C}$. Then $H = H_{\mathbb{C}}$ in this notation. We define the complex Lie group

$$G_{\mathbb{C}} = \{g \in GL(H_{\mathbb{C}}) |\ Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{C}}\},$$

and the real one

$$G_{\mathbb{R}} = \{g \in GL(H_{\mathbb{R}}) |\ Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{R}}\}.$$

We also have

$$G_{\mathbb{Z}} = \mathrm{Aut}(H_{\mathbb{Z}}, Q) = \{g \in GL(H_{\mathbb{Z}}) |\ Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{Z}}\}.$$

Griffiths in [9] showed that $G_{\mathbb{C}}$ acts on $\check{D}$ transitively, and so does $G_{\mathbb{R}}$ on D. The stabilizer of $G_{\mathbb{C}}$ on $\check{D}$ at the base point o is

$$B = \{g \in G_{\mathbb{C}} | gF^k_p = F^k_p,\ 0 \le k \le n\},$$

and the one of $G_{\mathbb{R}}$ on D is $V = B \cap G_{\mathbb{R}}$. Thus we can realize $\check{D}$, and D as

$$\check{D} = G_{\mathbb{C}}/B \ \text{ and } D = G_{\mathbb{R}}/V$$

so that $\check{D}$ is an algebraic manifold and $D \subseteq \check{D}$ is an open complex submanifold.

The Lie algebra $\mathfrak{g}$ of the complex Lie group $G_{\mathbb{C}}$ is

$$\mathfrak{g} = \{X \in \mathrm{End}(H_{\mathbb{C}}) |\ Q(Xu, v) + Q(u, Xv) = 0, \text{ for all } u, v \in H_{\mathbb{C}}\},$$

and the real subalgebra

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} |\ XH_{\mathbb{R}} \subseteq H_{\mathbb{R}}\}$$

is the Lie algebra of $G_{\mathbb{R}}$. Note that $\mathfrak{g}$ is a simple complex Lie algebra and contains $\mathfrak{g}_0$ as a real form, i.e., $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_0$.

On the linear space $\mathrm{Hom}(H_{\mathbb{C}}, H_{\mathbb{C}})$ we can put a Hodge structure of weight zero by

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^{k,-k} \quad \text{with} \quad \mathfrak{g}^{k,-k} = \{X \in \mathfrak{g} |\ XH^{r,n-r}_p \subseteq H^{r+k,n-r-k}_p,\ \forall r\}.$$

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By definition of B, the Lie algebra $\mathfrak{b}$ of B has the form $\mathfrak{b} = \bigoplus_{k\geq 0} \mathfrak{g}^{k,-k}$. Thus the Lie algebra $\mathfrak{v}_0$ of V is

$$\mathfrak{v}_0 = \mathfrak{g}_0 \cap \mathfrak{b} = \mathfrak{g}_0 \cap \mathfrak{b} \cap \overline{\mathfrak{b}} = \mathfrak{g}_0 \cap \mathfrak{g}^{0,0}.$$

Under these isomorphisms, the holomorphic tangent space of $\check{D}$ at the base point is naturally isomorphic to $\mathfrak{g}/\mathfrak{b}$.

Let us consider the nilpotent Lie subalgebra $\mathfrak{n}_+ := \oplus_{k\geq 1}\mathfrak{g}^{-k,k}$. $\mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}_+$. We denote the corresponding unipotent Lie group by

$$N_+ = \exp(\mathfrak{n}_+).$$

We proved the following basic propeties of the unipotent Lie group N_+ in [23].

Proposition 3.3. *(i) Let $o \in D$ be the base point. Then N_+ can be identified with its orbit $N_+(o)$ in $\check{D}$.*

(ii) The subset N_+ is an open complex submanifold in $\check{D}$, and $\check{D} \setminus N_+$ is an analytic subvariety of $\check{D}$ with codim$_{\mathbb{C}}(\check{D} \setminus N_+) \geq 1$.

Let $\mathcal{F}^k$, $0 \leq k \leq n$, be the Hodge bundles on D with fibers $\mathcal{F}^k|_s = F^k_s$ for any $s \in D$. Let $\mathcal{H}^{p,q} = \mathcal{F}^p/\mathcal{F}^{p+1}$, $p+q=n$, be the quotient bundles such that $\mathcal{H}^{p,q}|_s = H^{p,q}_s$. One can define the horizontal bundle $\mathrm{T}^{1,0}_h\check{D}$ in terms of the Hodge bundles $\mathcal{F}^k \to \check{D}$ by

$$\mathrm{T}^{1,0}_h\check{D} \simeq \mathrm{T}^{1,0}\check{D} \cap \bigoplus_{k=1}^{n} \mathrm{Hom}(\mathcal{F}^k/\mathcal{F}^{k+1}, \mathcal{F}^{k-1}/\mathcal{F}^k). \tag{3.8}$$

For the period map $\Phi : S \to \Gamma\backslash D$ and its lifting $\Phi : \mathcal{T} \to D$, we can pull back the Hodge bundles $\mathcal{F}^k \to D$, $0 \leq k \leq n$, to get the corresponding Hodge bundles on S and $\mathcal{T}$, which are still denoted by $\mathcal{F}^k \to S$ and $\mathcal{F}^k \to \mathcal{T}$, respectively, $0 \leq k \leq n$.

Remark 3.4. The canonical family $[\Omega^C(t)] = [e^{\Phi(t)} \lrcorner \Omega_0]$ in Theorem 2.3 is in fact a holomorphic section of the Hodge bundle $\mathcal{F}^n \simeq \mathcal{H}^{n,0}$ on Δ.

4. Canonical sections of the Hodge bundles for Calabi–Yau type manifolds

In this section, we extend the results in Section 2 to the case of Calabi–Yau type manifolds. Moreover, we get a finer formula than (4.1) by using the method of period maps in Hodge theory.

First let us recall the definition of Calabi–Yau type manifolds.

Definition 4.1. Let X be a projective manifold with $\dim_{\mathbb{C}} X = n$. We call X a Calabi–Yau type manifold if it satisfies the following conditions:

(i) There exists some integer k with $[n/2] < k \leq n$ such that

$$H^{\alpha,n-\alpha}_{\mathrm{pr}}(X) = 0$$

for $k < \alpha \leq n$, and $\dim_{\mathbb{C}} H^{k,n-k}_{\mathrm{pr}}(X) = 1$;

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(ii) For any generator $\Omega \in H^{k,n-k}_{\mathrm{pr}}(X)$, the contraction map

$$\lrcorner : H^1(X, \Theta_X) \to H^{k-1,n-k+1}_{\mathrm{pr}}(X), \qquad \phi \mapsto \phi \lrcorner \Omega$$

is an isomorphism.

The contraction $\phi \lrcorner \Omega$, for $\phi \in H^1(X, \Theta_X)$ and $\Omega \in H^{p,q}_{\mathrm{pr}}(X)$, is defined by

$$\phi \lrcorner \Omega \triangleq [\tilde{\phi} \lrcorner \tilde{\Omega}],$$

where $\tilde{\phi}$ is any representation of ϕ in $\mathrm{Z}^{0,1}(X, \mathrm{T}_X)$ via the isomorphism

$$H^1(X, \Theta_X) \simeq \frac{\mathrm{Z}^{0,1}(X, \mathrm{T}_X)}{\overline{\partial} A(X, \mathrm{T}_X)} \triangleq \frac{\{\overline{\partial}\text{-closed } (0,1)\text{-forms with values in } \mathrm{T}_X\}}{\{\overline{\partial}\text{-exact } (0,1)\text{-forms with values in } \mathrm{T}_X\}},$$

and $\tilde{\Omega}$ is any representative of Ω in the space of closed (p, q)-forms. Note that the notion is well-defined as a cohomology class $[\cdot]$.

By definition, we have the identification of the Hodge bundles

$$\mathcal{F}^k \simeq \mathcal{H}^{k,n-k}$$

for Calabi–Yau type manifolds.

In [23], we proved the following result.

Theorem 4.2. *Let X be a Calabi–Yau type manifold and let $f : \mathcal{X} \to \mathcal{T}$ be a family of Calabi–Yau type manifolds over a simply-connected complex manifold $\mathcal{T}$ containing $X_p \simeq X$ for $p \in \mathcal{T}$. Then there exists a Zariski open subset*

$$\check{\mathcal{T}} := \Phi^{-1}(N_+ \cap D)$$

of $\mathcal{T}$, on which we have the canonical holomorphic section Ω of the Hodge bundle $\mathcal{F}^k$. Moreover, in the canonical coordinate $\{U; z^c\}$ around the base point p (as in Definition 4.7), we have the following expansion of the holomorphic section $\Omega(z^c)$ of the Hodge bundle $\mathcal{F}^k$,

$$\Omega(z^c) = \Omega_0 + \sum_{1 \le i \le N} \theta^c_i \lrcorner \Omega_0 \cdot z^c_i + \sum_{1 \le i,j \le N} \theta^c_i \lrcorner \theta^c_j \lrcorner \Omega_0 \cdot z^c_i z^c_j + O(|z^c|^3), \tag{4.1}$$

where the higher order terms $O(|z^c|^3)$ lie in

$$\bigoplus_{\alpha \ge 2} H^{k-\alpha, n-k+\alpha}_{\mathrm{pr}}(X_p).$$

Here the section Ω, defined as the section of the Hodge bundle, is considered in the cohomology class, such that $\Omega(q) \in H^{k,n-k}_{\mathrm{pr}}(X_q)$ for $q \in \check{\mathcal{T}}$.

Remark 4.3. In the following, we will give the proof of Theorem 4.2 for the case of Calabi–Yau manifolds (i.e. $k = n$) of dimension $n = 3$, to illustrate the key ideas of the original proof in [23]. In fact, the reader can easily generalize the proof to higher dimension, and from Calabi–Yau manifolds to Calabi–Yau type manifolds by tensoring with the Hodge structure of Tate.

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Since the base manifold $\mathcal{T}$ of the family of Calabi–Yau type manifolds is simply-connected, we have a well-defined period map $\Phi : \mathcal{T} \to D$. We put

$$\check{\mathcal{T}} = \Phi^{-1}(N_+ \cap D).$$

By Proposition 3.3, we know that $\check{\mathcal{T}}$ is an open dense subset of $\mathcal{T}$, with $\mathcal{T} \setminus \check{\mathcal{T}}$ an analytic subset of $\mathcal{T}$.

In Definition 3.2, we have introduced the blocks of the adapted basis η of the Hodge decomposition at the base point p as

$$\eta = (\eta_{(0)}^T, \eta_{(1)}^T, \eta_{(2)}^T, \eta_{(3)}^T)^T,$$

where

$$\eta_{(\alpha)} = (\eta_{f^{-\alpha+3+1}}, \cdots, \eta_{f^{-\alpha+3}-1})^T$$

is the basis of $H_{\mathrm{pr}}^{3-\alpha,\alpha}(X_p)$ for $0 \leq \alpha \leq 3$.

For any $q \in \check{\mathcal{T}}$, we can choose the matrix representation of the image $\Phi(q)$ in N_+ by

$$\Phi(q) = (\Phi_{ij}(q))_{0 \leq i,j \leq m-1} \in N_+ \cap D.$$

Then, by Definition 3.2, the matrix $\Phi(q)$ is a block upper triangular matrix of the form,

$$\Phi(q) = \begin{bmatrix} 1 & \Phi^{(0,1)}(q) & \Phi^{(0,2)}(q) & \Phi^{(0,3)}(q) \\ 0 & I & \Phi^{(1,2)}(q) & \Phi^{(1,3)}(q) \\ 0 & 0 & I & \Phi^{(2,3)}(q) \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.2}$$

where, in the above notations, 0 denotes zero block matrix and I denotes identity block matrix, and the blocks $\Phi^{(0,1)}(q), \Phi^{(0,2)}(q), \Phi^{(0,3)}(q)$ are all row vectors.

By using the matrix representation, we have the adapted basis

$$\Omega(q) = (\Omega_{(0)}(q)^T, \Omega_{(1)}(q)^T, \Omega_{(2)}(q)^T, \Omega_{(3)}(q)^T)^T,$$

of the Hodge filtration at $q \in U$ as

$$\Omega_{(0)}(q) = \eta_{(0)} + \Phi^{(0,1)}(q) \cdot \eta_{(1)} + \Phi^{(0,2)}(q) \cdot \eta_{(2)} + \Phi^{(0,3)}(q) \cdot \eta_{(3)} \tag{4.3}$$

$$\Omega_{(1)}(q) = \eta_{(1)} + \Phi^{(1,2)}(q) \cdot \eta_{(2)} + \Phi^{(1,3)}(q) \cdot \eta_{(3)} \tag{4.4}$$

$$\Omega_{(2)}(q) = \eta_{(2)} + \Phi^{(2,3)}(q) \cdot \eta_{(3)}$$

$$\Omega_{(3)}(q) = \eta_{(3)}$$

where $\Omega_{(\alpha)}(q)$, together with $\Omega_{(0)}(q), \cdots, \Omega_{(\alpha-1)}(q)$, gives a basis of the Hodge filtration

$$F_q^{3-\alpha} = F^{3-\alpha} H_{\mathrm{pr}}^3(X_q, \mathbb{C}),$$

for $0 \leq \alpha \leq 3$.

We are interested in the section $\Omega_{(0)}(q)$ in (4.3) of the Hodge bundle $\mathcal{F}^3$. Note that, in equation (4.3), the term $\Phi^{(0,\beta)}(q) \cdot \eta_{(\beta)}$ is an element in $H_{\mathrm{pr}}^{3-\beta,\beta}(X_p)$, for $1 \leq \beta \leq 3$.

Let $q \in \check{\mathcal{T}}$ be any point. Let $\{U; z = (z_1, \cdots, z_N)\}$ be any holomorphic coordinate neighborhood around q with

$$z_\mu(q) = 0, \ 1 \leq \mu \leq N.$$

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In the following, the derivatives of the blocks,

$$\frac{\partial \Phi^{(\alpha,\beta)}}{\partial z_\mu}(z), \text{ for } 0 \le \alpha,\beta \le 3,\ 1 \le \mu \le N,$$

will denote the blocks of derivatives of its entries.

Lemma 4.4. *In the above notations,*

$$\left(\frac{\partial \Phi^{(0,2)}}{\partial z_\mu}(z), \frac{\partial \Phi^{(0,3)}}{\partial z_\mu}(z)\right) = \frac{\partial \Phi^{(0,1)}}{\partial z_\mu}(z)\cdot\left(\Phi^{(1,2)}(z), \Phi^{(1,3)}(z)\right) \tag{4.5}$$

$$\frac{\partial \Phi^{(1,3)}}{\partial z_\mu}(z) = \frac{\partial \Phi^{(1,2)}}{\partial z_\mu}(z)\cdot \Phi^{(2,3)}(z). \tag{4.6}$$

Here (4.5) *is equivalent to*

$$\frac{\partial \Phi^{(0,2)}}{\partial z_\mu}(z) = \frac{\partial \Phi^{(0,1)}}{\partial z_\mu}(z)\cdot \Phi^{(1,2)}(z) \tag{4.7}$$

and

$$\frac{\partial \Phi^{(0,3)}}{\partial z_\mu}(z) = \frac{\partial \Phi^{(0,1)}}{\partial z_\mu}(z)\cdot \Phi^{(1,3)}(z) \tag{4.8}$$

Proof. The main idea of the proof is to rewrite the Griffiths transversality in terms of the matrix representations of the image of the period map in $N_+ \cap D$. The rest only uses basic linear algebra.

By Griffiths transversality, especially the computations in page 813 of [10], we have that

$$\frac{\partial \Omega_{(\alpha)}}{\partial z_\mu}(z)$$

lies in $F_z^{2-\alpha}$, where

$$F_z^{2-\alpha} = F_{q'}^{2-\alpha} = F^{2-\alpha}H^3_{\mathrm{pr}}(X_{q'},\mathbb{C})$$

for $q' \in U$ around q with $z(q') = z$.

Then,

$$\frac{\partial \Omega_{(0)}}{\partial z_\mu}(z) = \frac{\partial \Phi^{(0,1)}}{\partial z_\mu}(z)\cdot\eta_{(1)} + \frac{\partial \Phi^{(0,2)}}{\partial z_\mu}(z)\cdot\eta_{(2)} + \frac{\partial \Phi^{(0,3)}}{\partial z_\mu}(z)\cdot\eta_{(3)} \in F_z^2, \tag{4.9}$$

which is spanned by $\Omega_{(0)}(z)$ and $\Omega_{(1)}(z)$. Therefore, there exists $A_{(0)}(z), A_{(1)}(z)$ such that

$$\begin{aligned}\frac{\partial \Omega_{(0)}}{\partial z_\mu}(z) &= A_{(0)}(z)\Omega_{(0)}(z) + A_{(1)}(z)\Omega_{(1)}(z)\\ &= A_{(0)}(z)\eta_{(0)} + (A_{(0)}(z)\Phi^{(0,1)}(z) + A_{(1)}(z))\cdot\eta_{(1)} + \cdots .\end{aligned} \tag{4.10}$$

By comparing the types in (4.9) and (4.10), we have that $A_{(0)}(z) = 0$, and

$$\begin{aligned}\frac{\partial \Omega_{(0)}}{\partial z_\mu}(z) &= A_{(1)}(z)\Omega_{(1)}(z)\\ &= A_{(1)}(z))\left(\eta_{(1)} + \Phi^{(1,2)}(z)\cdot\eta_{(2)} + \Phi^{(1,3)}(z)\cdot\eta_{(3)}\right)\end{aligned} \tag{4.11}$$

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By (4.9) and (4.11), we have

$$A_{(1)}(z)) = \frac{\partial \Phi^{(0,1)}}{\partial z_\mu}(z)$$

and

$$\begin{aligned}\frac{\partial \Omega_{(0)}}{\partial z_\mu}(z) &= \frac{\partial \Phi^{(0,1)}}{\partial z_\mu}(z)\Omega_{(1)}(z)\\ &= \frac{\partial \Phi^{(0,1)}}{\partial z_\mu}(z)\left(\eta_{(1)} + \Phi^{(1,2)}(z)\cdot\eta_{(2)} + \Phi^{(1,3)}(z)\cdot\eta_{(3)}\right),\end{aligned}$$

which, together with (4.9), implies (4.5).

Similarly, we have

$$\begin{aligned}\frac{\partial \Omega_{(1)}}{\partial z_\mu}(z) &= \frac{\partial \Phi^{(1,2)}}{\partial z_\mu}(z)\Omega_{(2)}(z)\\ &= \frac{\partial \Phi^{(1,2)}}{\partial z_\mu}(z)\cdot\eta_{(2)} + \frac{\partial \Phi^{(1,3)}}{\partial z_\mu}(z)\cdot\eta_{(3)}\\ &= \frac{\partial \Phi^{(1,2)}}{\partial z_\mu}(z)\left(\eta_{(2)} + \Phi^{(2,3)}(z)\cdot\eta_{(3)}\right),\end{aligned}$$

which implies (4.6). □

Remark 4.5. By the proof of Lemma 4.4, we have

$$(d\Phi)_q\left(\frac{\partial}{\partial z_\mu}\right) = \frac{\partial \Phi^{(0,1)}}{\partial z_\mu}(q) \oplus \frac{\partial \Phi^{(1,2)}}{\partial z_\mu}(q)$$

as elements in

$$\mathrm{Hom}(F_q^3, F_q^2/F_q^3) \oplus \mathrm{Hom}(F_q^2, F_q^1/F_q^2),$$

which maps $\Omega_{(0)}(q)$ and $\Omega_{(1)}(q)$ to

$$\frac{\partial \Phi^{(0,1)}}{\partial z_\mu}(q)\Omega_{(1)}(q) \text{ and } \frac{\partial \Phi^{(1,2)}}{\partial z_\mu}(q)\Omega_{(2)}(q)$$

respectively.

Proposition 4.6. *Let $\{U; z\}$ be a holomorphic coordinate around the base point p, such that $z_\mu(p) = 0$, $1 \le \mu \le N$, Then we have the following expansions of the blocks of $\Phi(z)$*

$$\begin{bmatrix} 1 & \Phi^{(0,1)}(z) & \Phi^{(0,2)}(z) & \Phi^{(0,3)}(z)\\ O & I & \Phi^{(1,2)}(z) & \Phi^{(1,3)}(z)\\ O & O & I & \Phi^{(2,3)}(z)\\ O & O & O & 1\end{bmatrix} = \begin{bmatrix} 1 & O(|z|) & O(|z|^2) & O(|z|^3)\\ O & I & O(|z|) & O(|z|^2)\\ O & O & I & O(|z|)\\ O & O & O & 1\end{bmatrix}. \tag{4.12}$$

Proof. At the base point p, the matrix representation in N_+ of $\Phi(p)$ is the identity matrix, and hence $\Phi^{(\alpha,\beta)}(0) = O$ for $\alpha < \beta$. This implies that the expansions of the blocks $\Phi^{(\alpha,\beta)}(z)$ have no constant terms, and that

$$\Phi^{(0,1)}(z), \Phi^{(1,2)}(z), \Phi^{(2,3)}(z) = O(|z|).$$

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By (4.6), we have

$$\frac{\partial\Phi^{(1,3)}}{\partial z_\mu}(z)=\frac{\partial\Phi^{(1,2)}}{\partial z_\mu}(z)\cdot\Phi^{(2,3)}(z)=O(|z|),$$

which implies that $\Phi^{(1,3)}(z)=O(|z|^2)$, since $\Phi^{(1,3)}(z)$ have no constant terms. Similarly, (4.7) implies that

$$\frac{\partial\Phi^{(0,2)}}{\partial z_\mu}(z)=\frac{\partial\Phi^{(0,1)}}{\partial z_\mu}(z)\cdot\Phi^{(1,2)}(z)=O(|z|)$$

which in turn implies that $\Phi^{(0,2)}(z)=O(|z|^2)$.

Finally, from (4.8), we have

$$\frac{\partial\Phi^{(0,3)}}{\partial z_\mu}(z)=\frac{\partial\Phi^{(0,1)}}{\partial z_\mu}(z)\cdot\Phi^{(1,3)}(z)=O(|z|^2)$$

which implies that $\Phi^{(0,3)}(z)=O(|z|^3)$. □

Now we are ready to prove Theorem 4.2 for the case of Calabi–Yau manifolds of dimension $n=3$.

First, we give the definition of canonical coordinates on $\check{\mathcal{T}}$. Let $q\in\check{\mathcal{T}}$ be any point. Let $\{U;z=(z_1,\cdots,z_N)\}$ be any holomorphic coordinate neighborhood around q with

$$z_\mu(q)=0,\ 1\le\mu\le N.$$

By Condition (ii) in Definition 4.1, we have that

$$(d\Phi)_q:\ \mathrm{T}_qU\to\mathrm{Hom}(F_q^3,F_q^2/F_q^3)$$

is an isomorphism. Then, by Remark 4.5, the matrix

$$\frac{\partial\Phi^{(0,1)}}{\partial z}(0)\in\mathrm{Hom}(F_q^3,F_q^2/F_q^3)$$

is non-degenerate. We define

$$z^c=(z_1^c(z),\cdots,z_N^c(z))=\Phi^{(0,1)}(z),\tag{4.13}$$

which is also holomorphic coordinate in a neighborhood around q by shrinking U if necessary.

Definition 4.7. We call the coordinate $\{U;z^c=(z_1^c,\cdots,z_N^c)\}$, defined by the equations (4.13), the canonical coordinate.

Proof of Theorem 4.2. Write

$$\Omega_0=\eta_{(0)}\in H^{n,0}_{\mathrm{pr}}(X_p)$$

and $\Omega(q)=\Omega_{(0)}(q)$ as given in (4.3).

By Definition 4.7, we have

$$\begin{aligned}\Omega(z^c)&:=\Omega_{(0)}(z^c)\\&=\eta_{(0)}+z^c\cdot\eta_{(1)}+\Phi^{(0,2)}(z^c)\cdot\eta_{(2)}+\Phi^{(0,3)}(z^c)\cdot\eta_{(3)}\\&=\Omega_0+\sum_{1\le i\le N}\eta_{f^{n-1}+i-1}z_i^c+\Phi^{(0,2)}(z^c)\cdot\eta_{(2)}+\Phi^{(0,3)}(z^c)\cdot\eta_{(3)}\end{aligned}\tag{4.14}$$

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By the proof of Griffiths transversality in pages 813–814 of [10], we know that

$$\eta_{f^{n-1}+i-1} = \frac{\partial \Omega}{\partial z_i^c}(0) = \theta_i^c \lrcorner \Omega_0, \tag{4.15}$$

where

$$\theta_i^c = \rho\left(\frac{\partial}{\partial z_i^c}\right) \in H^1(X_p, \Theta_{X_p})$$

is the image of the Kodaira-Spencer map in the canonical coordinate z^c.

Applying Proposition 4.6 for the canonical coordinate, we have

$$\Phi^{(0,2)}(z^c) \cdot \eta_{(2)} = O(|z^c|^2) \in H^{1,2}_{\mathrm{pr}}(X_p), \tag{4.16}$$

and

$$\Phi^{(0,3)}(z^c) \cdot \eta_{(3)} = O(|z^c|^3) \in H^{0,3}_{\mathrm{pr}}(X_p). \tag{4.17}$$

Therefore, in order to prove that

$$\Omega(z^c) = \Omega_0 + \sum_{1\le i\le N} \theta_i^c \lrcorner \Omega_0 \cdot z_i^c + \sum_{1\le i,j\le N} \theta_i^c \lrcorner \theta_j^c \lrcorner \Omega_0 \cdot z_i^c z_j^c + O(|z^c|^3),$$

we only need to check that the third term in equation (4.14) has the following form,

$$\Phi^{(0,2)}(z^c) \cdot \eta_{(2)} = \sum_{1\le i,j\le N} \theta_i^c \lrcorner \theta_j^c \lrcorner \Omega_0 \cdot z_i^c z_j^c + O(|z^c|^3). \tag{4.18}$$

In fact, equations (4.16) and (4.17) imply that the second order term $\frac{\partial^2 \Omega}{\partial z_i^c \partial z_j^c}(0)$ of the expansion of $\Omega(z^c)$ lies in $H^{1,2}_{\mathrm{pr}}(X_p)$. By the calculation in page 813 of [10], the second order term

$$\frac{\partial^2 \Omega}{\partial z_i^c \partial z_j^c}(0) = \theta_{ij}^c \lrcorner \Omega_0 + \theta_i^c \lrcorner \theta_j^c \lrcorner \Omega_0,$$

where θ_{ij}^c is the second order term of the expansion of $\phi(z^c)$. Here

$$\phi(z^c) = \sum_i \theta_i^c z_i^c + \sum_{ij} \theta_{ij}^c z_i^c z_j^c + O(|z^c|^3) \in A^{0,1}(X_p, \mathrm{T}^{1,0} X_p),$$

is the Beltrami differential which defines the complex structures on the polarized manifolds near p.

By comparing types, we have

$$\theta_{ij}^c \lrcorner \Omega_0 = \frac{\partial^2 \Omega}{\partial z_i^c \partial z_j^c}(0) - \theta_i^c \lrcorner \theta_j^c \lrcorner \Omega_0 \in H^{2,1}_{\mathrm{pr}}(X_p) \cap H^{1,2}_{\mathrm{pr}}(X_p) = 0,$$

which implies that

$$\frac{\partial^2 \Omega}{\partial z_i^c \partial z_j^c}(0) = \theta_i^c \lrcorner \theta_j^c \lrcorner \Omega_0,$$

under canonical coordinates z^c. this verifies equation (4.18), and completes the proof. □

Remark 4.8. From the proof of Theorem 4.2, we see that the canonical coordinates, defined by certain blocks of the matrices representing the image of the period map, is of crucial importance to the expansion formula (4.1). Roughly speaking, the canonical coordinates annihilate the component $\theta_{ij}^c \lrcorner \Omega_0$ of the second order term $\frac{\partial^2 \Omega}{\partial z_i^c \partial z_j^c}(0)$ in $H^{2,1}(X_p)$, which is not necessarily zero for general coordinates around the base point.

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5. Applications to moduli spaces as ball quotients

Now let us introduce the definition of polarized manifolds of ball type, which is used to characterize the corresponding period domain admitting a complex ball as sub-domain.

Definition 5.1. The polarized manifold (X, L) is said to be of ball type if the following conditions are satisfied:

(i) There exists some integer k with $[n/2] < k \leq n$ such that

$$H^{\alpha,n-\alpha}_{\mathrm{pr}}(X) = 0$$

for $k < \alpha \leq n$.

(ii) There is an element Ω in $H^{k,n-k}_{\mathrm{pr}}(X)$ such that, if $\{\theta_i\}_{1\leq i\leq N}$ is the basis of $H^1(X, \Theta_X)$, then

$$(*_1) \qquad \{\theta_i \lrcorner \Omega\}_{1\leq i\leq N} \text{ is linearly independent in } H^{k-1,n-k+1}_{\mathrm{pr}}(X),$$

and

$$(*_2) \qquad \theta_i \lrcorner \theta_j \lrcorner (H^{k,n-k}_{\mathrm{pr}}(X)) = 0, \text{ for any } 1 \leq i, j \leq N.$$

We refer the reader to [23] for the motivation behind the above conditions.

Theorem 5.2. *Let X be the polarized manifold of ball type with $\Omega_0 \in H^{k,n-k}_{\mathrm{pr}}(X)$. Then, in the canonical coordinate $\{U; z^c\}$ around the base point p, we have the canonical holomorphic canonical section Ω of the Hodge bundle $\mathcal{F}^k$ such that $\Omega(z^c)$ satisfies $(*_1)$ in Definition 5.1, with following expansion*

$$\Omega(z^c) = \Omega_0 + \sum_{1\leq i\leq N} \theta^c_i \lrcorner \Omega_0 \cdot z^c_i. \tag{5.1}$$

In this review, to make our idea more clear, we shall only prove Theorem 5.2 for polarized manifolds of ball types under stronger conditions than those in Definition 5.1, which is stated as follows.

Let $f : \mathcal{X} \to S$ of polarized manifolds ball type containing X over the connected complex manifold S. Then

$$S \ni q \mapsto H^n_{\mathrm{pr}}(X_q, \mathbb{C})$$

is a local system. We assume that there exists a subgroup $H^n_0(X_q, \mathbb{C}) \subset H^n_{\mathrm{pr}}(X_q, \mathbb{C})$ for any $q \in S$ such that

$$S \ni q \mapsto H^n_0(X_q, \mathbb{C})$$

is also a local system, and moreover, if we denote $H^{\alpha,n-\alpha}_0(X) = H^{\alpha,n-\alpha}_{\mathrm{pr}}(X) \cap H^n_0(X_q, \mathbb{C})$, then

$(*'_1)$ $\dim_{\mathbb{C}} H^{k,n-k}_0(X) = 1$ with generator Ω as given in Definition 5.1 such that the contraction map

$$\lrcorner : H^1(X, \Theta_X) \to H^{k-1,n-k+1}_0(X), \qquad \phi \mapsto \phi \lrcorner \Omega$$

is an isomorphism.

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The above assumption simplifies the proof, yet makes the key points in the original proof in [23] apparent.

Let us prove Theorem 5.2 for the special case that $n=3$ and $k=n$ in Definition 5.1.

Proof of Theorem 5.2. Let D be the period domain for polarized manifolds of ball type. From $(*_1')$ we know that D has a sub-domain D_0 consisting of the Hodge decompositions

$$H_0^n(X,\mathbb{C}) = H_0^{3,0}(X)\oplus H_0^{2,1}(X)\oplus H_0^{1,2}(X)\oplus H_0^{0,3}(X),$$

which is isomorphic to the period domain for Calabi–Yau type manifolds. Then one can construct the refined period map

$$\Phi_0 :\ S\to \Gamma_0\backslash D_0$$

from the base S of the family $f:\mathcal{X}\to S$ of polarized manifolds ball type, where Γ_0 is the subgroup of the monodromy group Γ preserving D_0.

Therefore, we can apply Theorem 4.2 to construct the canonical holomorphic canonical section Ω of the Hodge bundle $\mathcal{F}^3$ such that $\Omega(z^c)$ satisfies $(*_1')$ in Definition 5.1, with following expansion

$$\begin{aligned}\Omega(z^c) &= \Omega_0+\sum_{1\le i\le N}\theta_i^c\lrcorner\Omega_0\cdot z_i^c+\sum_{1\le i,j\le N}\theta_i^c\lrcorner\theta_j^c\lrcorner\Omega_0\cdot z_i^c z_j^c+O(|z^c|^3)\\ &= \Omega_0+\sum_{1\le i\le N}\theta_i^c\lrcorner\Omega_0\cdot z_i^c+O(|z^c|^3),\end{aligned}\tag{5.2}$$

where $O(|z^c|^3)\in H^{1,2}_{\mathrm{pr}}(X_p)\oplus H^{0,3}_{\mathrm{pr}}(X_p)$, where the last equality follows from $(*_2)$

By the calculation in page 813 of [10] and equation $(*_2)$ again, for z^c near the origin, the second order term at z^c is

$$\frac{\partial^2\Omega}{\partial z_i^c\partial z_j^c}(z^c)=\theta_i^c(z^c)\lrcorner\theta_j^c(z^c)\lrcorner\Omega(z^c)+\theta_{ij}^c(z^c)\lrcorner\Omega(z^c)=\theta_{ij}^c(z^c)\lrcorner\Omega(z^c),$$

where the $\theta_i^c(z^c)$'s are the first terms of the local expansion of the Beltrami differential at the point z^c, and the $\theta_{ij}^c(z^c)$'s are the second terms.

Note that

$$\theta_{ij}^c(z^c)\lrcorner\Omega(z^c)\in F^2_{z^c}/F^3_{z^c},$$

which, combined with the basis given by equation (4.4), gives

$$\begin{aligned}\theta_{ij}^c(z^c\lrcorner\Omega(z^c)) &= A(z^c)\cdot\Omega_{(1)}\\ &= A(z^c)\cdot\Big(\eta_{(1)}+\Phi^{(1,2)}(z^c)\cdot\eta_{(2)}+\Phi^{(1,3)}(z^c)\cdot\eta_{(3)}\Big),\end{aligned}$$

for some matrix $A(z^c)$.

By equation (5.2), we have

$$\frac{\partial^2\Omega}{\partial z_i^c\partial z_j^c}(z^c)=\frac{\partial^2}{\partial z_i^c\partial z_j^c}O(|z^c|^3)(\Omega)\in H^{1,2}_{\mathrm{pr}}(X_p)\oplus H^{0,3}_{\mathrm{pr}}(X_p).$$

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Therefore,

$$\begin{aligned} A(z^c)\cdot\eta_{(1)} &= \frac{\partial^2\Omega}{\partial z_i^c\partial z_j^c}(z^c) - A(z^c)\cdot\left(\Phi^{(1,2)}(z^c)\cdot\eta_{(2)}+\Phi^{(1,3)}(z^c)\cdot\eta_{(3)}\right) \\ &\in H^{2,1}_{\mathrm{pr}}(X_p)\cap\left(H^{1,2}_{\mathrm{pr}}(X_p)\oplus H^{0,3}_{\mathrm{pr}}(X_p)\right)=0, \end{aligned}$$

which implies that $A(z^c)=0$, and so $\theta^c_{ij}(z^c)\lrcorner\Omega(z^c)=0$ for any z^c near the origin. Thus

$$\frac{\partial^2\Omega}{\partial z_i^c\partial z_j^c}(z^c)=0$$

for any z^c near the origin. This implies that the higher order terms $O(|z^c|^3)(\Omega)$ in equation (5.2) vanish, which gives us the expansion (5.1). □

Now let $f:\mathcal{X}\to S$ be a family of polarized manifolds ball type. Let

$$\Phi_0:\ S\to\Gamma_0\backslash D_0$$

be the refined period map and

$$\Phi_0:\ \mathcal{T}\to D_0$$

be the lifted period map from the universal cover $\mathcal{T}$ of S. Let

$$\check{\mathcal{T}}=\Phi^{-1}(N_+\cap D_0).$$

Then, by Proposition 3.3, $\mathcal{T}\setminus\check{\mathcal{T}}$ is an analytic subset of $\mathcal{T}$. In Definition 4.7, the canonical coordinate

$$z^c(q)=(z_1^c(q),\cdots,z_N^c(q))=\Phi^{(0,1)}(q),$$

$q\in\check{\mathcal{T}}$, is globally defined on $\check{\mathcal{T}}$.

Now we take the adapted basis η of the Hodge decomposition at the base point p an orthogonal basis with the Hodge metric. Then, by Theorem 5.2 and Hodge-Riemann bilinear relations, we have

$$-1+\sum_{1\le i\le N}|z_i^c(q)|^2<0,$$

which is equivalent to

$$z^c(q)=\Phi^{(0,1)}(q)\in\mathbb{B}^N,$$

with $\mathbb{B}^N$ the complex ball defined by

$$\mathbb{B}^N:=\left\{w\in\mathbb{C}^N:\ \sum_{1\le i\le N}|w_i|^2<1\right\}.$$

By Riemann extension theorem, $z^c(q)=\Phi^{(0,1)}(q)$ is globally defined on $q\in\mathcal{T}$. This is equivalent to the refined period domain D_0 having a sub-domain

$$\{\mathbb{C}\{\Omega\}\subseteq H_0^n(X,\mathbb{C}):\ \Omega\in H_0^{k,n-k},\ H_0^n(X,\mathbb{C})=\oplus_{p+q=n}H_0^{p,q}\}$$

which is isomorphic to the complex ball $\mathbb{B}^N$, such that the composite holomorphic map

$$\wp:\ \mathcal{T}\to\mathbb{B}^N$$

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of Φ with the projection

$$P:\ D_0 \to \mathbb{B}^N,$$
$$(H_0^n(X,\mathbb{C}) = \oplus_{p+q=n} H_0^{p,q}) \mapsto \mathbb{C}\{\Omega\}$$

for $\Omega \in H_0^{k,n-k}$, is locally an isomorphism. Here $\mathbb{C}\{\Omega\}$ denotes the complex line generated by Ω.

Corollary 5.3. *Let $\mathcal{M}$ be the moduli space of polarized manifolds of ball type. Then $\mathcal{M}$ admits a locally isomorphic holomorphic map to the ball quotient $\Gamma_B \backslash \mathbb{B}^N$, via the the period map*

$$\wp:\ \mathcal{M} \to \Gamma_B \backslash \mathbb{B}^N,$$

where Γ_B denotes the subgroup of the monodromy group Γ preserving $\mathbb{B}^N$.

There are many interesting examples of moduli spaces of ball type.

Example 5.4. Let $\mathcal{M}$ be the moduli space of polarized manifolds satisfying the following conditions:

(1) $\mathcal{M}$ is the moduli space of cubic surfaces or cubic threefolds;

(2) $\mathcal{M}$ admits an analytic family $g:\mathcal{U} \to \mathcal{M}_0$ with $\mathcal{M}_0$ the Zariski open subset of $\mathcal{M}$ or some covering space of $\mathcal{M}$. Moreover, there is a finite abelian group G acting holomorphically on $\mathcal{U}$ and there exists some $\chi \in \mathrm{Hom}(G,\mathbb{C}^*)$, such that the corresponding eigen-spaces F_χ^j of $F^jH^n(X,\mathbb{C})$, $o \le j \le n$ satisfy

$$\begin{aligned}
&\text{(i). } F_\chi^j = 0, \text{ for } j \ge k+1;\\
&\text{(ii). } \dim_{\mathbb{C}}(F_\chi^k) = 1;\\
&\text{(iii). } H^1(X,\Theta_X) \xrightarrow{\simeq} \mathrm{Hom}(F_\chi^k, F_\chi^{k-1}/F_\chi^k);\\
&\text{(iv). } F_\chi^j = F_\chi^{k-1}, \text{ for } j \le k-1,
\end{aligned}$$

(3) (Deligne-Mostow theory in [7]) $\mathcal{M}$ is the moduli spacs of the arrangements of m points $p_1, \cdots p_m$ in $\mathbb{P}^1$

Then $\mathcal{M}$ admits a locally isomorphic holomorphic map to the ball quotient $\Gamma_B \backslash \mathbb{B}^N$.

See [23] and the references therein for details on the above examples. Furthermore, we can prove that these local isomorphic holomorphic maps are actually global biholomorphic maps.

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