Journal of the Iranian Mathematical Society
ISSN (on-line): 2717-1612
J. Iran. Math. Soc. 5 (2024), no. 2, 79-93

CHARACTERIZATION OF THE STRUCTURED PSEUDOSPECTRUM IN NON-ARCHIMEDEAN BANACH SPACES

J. ETTAYB

Abstract. In this paper, we demonstrate some results on the pseudospectrum of linear operator pencils on non-Archimedean Banach spaces. In particular, we give a relationship between the Fredholm spectrum of a bounded operator pencil (A, B) and the Fredholm spectrum of the pencil (A^{-1}, B^{-1}). Also, we characterize the essential spectrum of operator pencils on non-Archimedean Banach spaces. Furthermore, we introduce and study the structured pseudospectrum of linear operators on non-Archimedean Banach spaces. We prove that the structured pseudospectra associated with various ε are nested sets, and the intersection of all the structured pseudospectra is the spectrum. We characterize the structured pseudospectrum of bounded linear operators on non-Archimedean Banach spaces. Finally, we characterize the structured essential pseudospectrum of bounded linear operator pencils on non-Archimedean Banach spaces and we give an illustrative example.

1. Introduction

In the classical setting, spectral theory has witnessed an explosive development by many researchers who have presented a survey of results concerning various types of essential spectrum and pseudospectrum [7, 15, 19]. Recently, Davies [7] introduced the concept of structured pseudospectrum of linear operators on a complex Banach space. Moreover, Abdmouleh, Ammar,and Jeribi [1] gave a characterization of the S-essential spectrum and defined the S-Riesz projection. On the other hand, they investigated the S-Browder resolvent and studied the S-essential spectrum of the sum of two bounded linear operators acting on a complex Banach space.

Communicated by Massoud M. Amini
MSC(2020): Primary: 47S10; Secondary: 47A10; 47A53.
Keywords: Non-Archimedean Banach spaces, pseudospectrum, condition pseudospectrum, linear operator pencils.
Received: 1 March 2024, Accepted: 15 April 2024.
DOI: https://dx.doi.org/10.30504/JIMS.2024.446376.1163

The non-Archimedean Banach spaces were studied by Monna [16] which played a central role in non-Archimedean functional analysis. There are many differences between non-Archimedean Banach spaces and classical cases, see [3, 6, 14, 16, 17]. One of the main purposes of non-Archimedean Banach spaces is to study the non-Archimedean operator theory and spectral theory.

In non-Archimedean operator theory, Ammar, Bouchekoua and Jeribi [2] introduced and studied the pseudospectrum and the essential pseudospectrum of linear operators on a non-Archimedean Banach space and the non-Archimedean Hilbert space E_ω, respectively. In particular, they characterized these pseudospectrum. Furthermore, inspired by Diagana and Ramaroson [6], they established a relationship between the essential pseudospectrum of a closed linear operator and the essential pseudospectrum of this closed linear operator perturbed by completely continuous operators on the non-Archimedean Hilbert space E_ω. Moreover, Ettayb [10] introduced and studied the bounded linear operator pencils, the pseudospectrum, and the essential pseudospectrum of bounded linear operator pencils on non-Archimedean Banach spaces. Furthermore, Blali, El Amrani, and Ettayb [5] gave a characterization of the essential spectrum of the operator pencil (A, B), where A is a closed linear operator and B is a bounded linear operator through the Fredholm operators on a Banach space of countable type over $\mathbb{Q}_p$. In [4], Blali, El Amrani, and Ettayb defined and studied the trace pseudospectrum, the ε-determinant spectrum, and the ε-trace of bounded linear operator pencils on non-Archimedean Banach spaces. Recently, Ettayb [11] defined and established some results on the C-trace pseudospectrum, the M-determinant pseudospectrum and the pseudospectrum of non-Archimedean matrix pencils. This work is motivated by many studies related to the topic of eigenvalue problems in non-Archimedean operator theory and perturbation theory, see [2, 3, 5, 13, 17].

The purpose of this work is to prove more results on the non-Archimedean pseudospectrum of operator pencils. We initiate the study of non-Archimedean structured pseudospectrum of linear operators.

Throughout this paper, X and Y are non-Archimedean Banach spaces over a complete non-Archimedean valued field $\mathbb{K}$ with a non-trivial valuation $|\cdot|$, $\mathcal{L}(X, Y)$ denotes the set of all bounded linear operators from X into Y and $X^* = \mathcal{L}(X, \mathbb{K})$ is the dual space of X. When $X = Y$, we set $\mathcal{L}(X, Y) = \mathcal{L}(X)$. Let $A \in \mathcal{L}(X)$, $N(A)$ and $R(A)$ denote the kernel and range of A respectively. For additional details, we refer to [6, 18]. The space X is said to be spherically complete if the intersection of every decreasing sequence of balls in X is nonempty. Recall that, an unbounded linear operator $A : D(A) \subseteq X \to Y$ is said to be closed if for all $(x_n)_{n\in\mathbb{N}} \subset D(A)$ such that $\|x_n - x\| \to 0$ and $\|Ax_n - y\| \to 0$ as $n \to \infty$, for some $x \in X$ and $y \in Y$, then $x \in D(A)$ and $y = Ax$. The collection of all closed linear operators from X into Y is denoted by $\mathcal{C}(X, Y)$. When $X = Y$, we put $\mathcal{C}(X, X) = \mathcal{C}(X)$. Note that, if $A \in \mathcal{L}(X)$ and B is an unbounded linear operator, then $A + B$ is closed if and only if B is closed [6]. We refer to [3, 6, 18] for more details on non-Archimedean operator theory. There are many interesting works on pseudospectrum in the classical Banach spaces, see [15, 19].

DOI: https://dx.doi.org/10.30504/JIMS.2024.446376.1163

2. Preliminaries

In the next definition, X and Y are two vector spaces over $\mathbb{K}$.

Definition 2.1 ([17]). *We say that $A \in \mathcal{L}(X,Y)$ has an index, when both $\alpha(A) = \dim N(A)$ and $\beta(A) = \dim\Big(Y/R(A)\Big)$ are finite. In this case, the index of the linear operator A is defined as $ind(A) = \alpha(A) - \beta(A)$.*

Definition 2.2 ([17]). *Let $A \in \mathcal{L}(X,Y)$, A is said to be upper semi-Fredholm operator, if $\alpha(A)$ is finite and $R(A)$ is closed. The set of all upper semi-Fredholm operators from X into Y is denoted by $\Phi_+(X,Y)$.*

Definition 2.3 ([17]). *Let $A \in \mathcal{L}(X,Y)$, A is said to be lower semi-Fredholm operator, if $\beta(A)$ is finite. The set of all lower semi-Fredholm operators from X into Y is denoted by $\Phi_-(X,Y)$.*

The set of all Fredholm operators from X into Y is defined by

$$\Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y).$$

Let X be a non-Archimedean Banach space over $\mathbb{K}$. A subset A of X is said to be compactoid, if for every $\varepsilon > 0$, there is a finite subset B of X such that $A \subset B_\varepsilon(0) + C_0(B)$, where $B_\varepsilon(0) = \{x \in X : \|x\| \leq \varepsilon\}$ and $C_0(B)$ is the absolutely convex hull of X, i.e.,

$$C_0(B) = \{\lambda_1 x_1 + \cdots + \lambda_n x_n : n \in \mathbb{N},\ \lambda_1, \cdots, \lambda_n \in B_{\mathbb{K}},\ x_1, \cdots, x_n \in B\}.$$

For additional details, see [18]. Now, we recall the notions of compact operators, operators of finite rank and completely continuous operators.

Definition 2.4 ([18]). *Let $A \in \mathcal{L}(X,Y)$. A is said to be compact, if $A(B_X)$ is compactoid in Y, where $B_X = \{x \in X : \|x\| \leq 1\}$.*

We denote by $\mathcal{K}(X,Y)$, the set of all compact operators from X into Y.

Definition 2.5 ([18]). *Let $A \in \mathcal{L}(X,Y)$. A is called an operator of finite rank, if $\dim R(A)$ is finite. The set of all operators of finite rank is denoted by $\mathcal{F}_0(X,Y)$.*

Definition 2.6 ([6]). *Let X be a non-Archimedean Banach space over $\mathbb{K}$ and let $A \in \mathcal{L}(X)$. A is said to be completely continuous, if there exists a sequence $(A_n)_n$ in $\mathcal{F}_0(X)$ such that $\|A_n - A\| \to 0$ as $n \to \infty$. The collection of completely continuous linear operators on X is denoted by $\mathcal{C}_c(X)$.*

Now, we give a characterization of compact operators as follows.

Theorem 2.7 ([18]). *Let $A \in \mathcal{L}(X,Y)$. Then A is compact if, and only if, for every $\varepsilon > 0$, there exists an operator $S \in \mathcal{L}(X,Y)$ such that $R(S)$ is finite-dimensional and $\|A - S\| < \varepsilon$.*

Remark 2.8 ([18]).

DOI: https://dx.doi.org/10.30504/JIMS.2024.446376.1163

(i) *In a non-Archimedean Banach space X, we do not have the relationship between $\mathcal{C}_c(X)$ and $\mathcal{K}(X)$ as a classical case. Serre has proved that those concepts coincide, when $\mathbb{K}$ is locally compact.*
(ii) *If $\mathbb{K}$ is locally compact. Then all completely continuous linear operators on X are compact.*
(iii) *If $\mathbb{K}$ is locally compact. Then A is compact if, and only if, $A(B_X)$ has compact closure.*

The following theorem showed that the set of all Fredholm operators is invariant under preservation by compact operators.

Theorem 2.9 ([17])**.** *Suppose that $\mathbb{K}$ is spherically complete. Then, for each $A \in \Phi(X, Y)$ and $K \in \mathcal{K}(X, Y)$, $A + K \in \Phi(X, Y)$ and $ind(A + K) = ind(A)$.*

Lemma 2.10 ([13])**.** *Suppose that $\mathbb{K}$ is spherically complete. If $x_1^*, \cdots, x_n^*$ are linearly independent vectors in X^*, then there are vectors $x_1, \cdots, x_n$ in X such that*

$$x_j^*(x_k) = \delta_{j,k} = \begin{cases} 1, \ if \ j = k; \\ 0, \ if \ j \neq k. \end{cases} \quad 1 \leq j, k \leq n. \tag{2.1}$$

Moreover, if $x_1, \cdots, x_n$ are linearly independent vectors in X, then there are vectors $x_1^*, \cdots, x_n^*$ in X^* such that (2.1) holds.

Theorem 2.11 ([14])**.** *Assume that X, Y are non-Archimedean Banach spaces over $\mathbb{K}$. Let $A : D(A) \subseteq X \to Y$ be a surjective closed linear operator. Then A is an open map.*

When the domain of A is dense in X, the adjoint operator A^* of A is defined as usual. Specifically, the operator $A^* : D(A^*) \subseteq Y^* \to X^*$ satisfies

$$\langle Ax, \ y^* \rangle = \langle x, \ A^* y^* \rangle$$

for all $x \in D(A)$, $y^* \in D(A^*)$.

Theorem 2.12 ([18])**.** *Suppose that $\mathbb{K}$ is spherically complete. Let X be a non-Archimedean Banach space over $\mathbb{K}$. For any $x \in X \backslash \{0\}$, there exists $x^* \in X^*$ such that $x^*(x) = 1$ and $\|x^*\| = \|x\|^{-1}$.*

Remark 2.13 ([18])**.** *$\mathbb{Q}_p$ is spherically complete and locally compact.*

In the next theorem, $\Phi_0(X, Y)$ denotes the set of all bounded linear Fredholm operators of index zero.

Theorem 2.14 ([13])**.** *Let $\mathbb{K}$ be spherically complete. Let X, Y be non-Archimedean Banach spaces over $\mathbb{K}$. Every operator in $\Phi_0(X, Y)$ is a sum of an invertible operator and an operator of finite rank.*

Corollary 2.15 ([13])**.** *If X, Y are non-Archimedean Banach spaces over $\mathbb{Q}_p$ and $B \in \mathcal{L}(X, Y)$ where B is invertible and K is compact, then $ind(B + K) = 0$.*

Theorem 2.16 ([13])**.** *Let X be a non-Archimedean Banach space over a spherically complete field $\mathbb{K}$. If $A, B \in \Phi(X)$, then $BA \in \Phi(X)$.*

DOI: https://dx.doi.org/10.30504/JIMS.2024.446376.1163

Theorem 2.17 ([10]). *Let X be a non-Archimedean Banach space over a spherically complete field $\mathbb{K}$ such that $\|X\| \subseteq |\mathbb{K}|$, and let $A, B \in \mathcal{L}(X)$ and $\varepsilon > 0$. Then,*

$$\sigma_\varepsilon(A,B) = \bigcup_{C \in \mathcal{L}(X): \|C\| < \varepsilon} \sigma(A+C, B).$$

3. Main Results

From Theorem 2.14, we conclude the following lemma.

Lemma 3.1. *Let X, Y be non-Archimedean Banach spaces over $\mathbb{Q}_p$. Every operator in $\Phi_0(X,Y)$ is a sum of an invertible operator and compact operator.*

We have the following proposition.

Proposition 3.2. *Let X, Y be non-Archimedean Banach spaces over $\mathbb{Q}_p$. Then $A \in \Phi_0(X,Y)$ if and only if $A = B + K$ where B is invertible and K is compact.*

Proof. Let $A \in \Phi_0(X,Y)$. By Theorem 2.14, $A = B + K$ where B is invertible and K is of finite rank. Since $\mathbb{K} = \mathbb{Q}_p$, by Theorem 2.7, K is a compact operator. The converse follows from Corollary 2.15. □

As the classical setting, we have the following lemma.

Lemma 3.3. *Let X be a non-Archimedean Banach space over $\mathbb{Q}_p$. Suppose that $A \in \mathcal{L}(X)$ and there are $B_0, B_1 \in \mathcal{L}(X)$ such that B_0A and AB_1 are in $\Phi(X)$. Then $A \in \Phi(X)$.*

Definition 3.4. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $A, B \in \mathcal{L}(X)$. The Fredholm spectrum $\sigma_F(A,B)$ of the operator pencil (A,B) of the form $A - \lambda B$ is given by*

$$\sigma_F(A,B) = \{\lambda \in \mathbb{K} : A - \lambda B \notin \Phi(X)\}.$$

The Fredholm resolvent of (A,B) is $\rho_F(A,B) = \mathbb{K} \backslash \sigma_F(A,B)$.

The following theorem gives a relationship between the Fredholm spectrum of a bounded operator pencil (A,B) and the Fredholm spectrum of the operator pencil (A^{-1}, B^{-1}).

Theorem 3.5. *Let X be non-Archimedean Banach space over a spherically complete field $\mathbb{K}$, and let $A, B \in \mathcal{L}(X)$ such that $AB = BA$ and $0 \in \rho(A) \cap \rho(B)$. Then $\lambda \in \sigma_F(A,B)$ if and only if $\frac{1}{\lambda} \in \sigma_F(A^{-1}, B^{-1})$.*

Proof. We have

$$A - \lambda B = -\lambda B(A^{-1} - \lambda^{-1}B^{-1})A. \tag{3.1}$$

Let $\frac{1}{\lambda} \in \mathbb{K} \backslash \sigma_F(A^{-1}, B^{-1})$, then $A^{-1} - \lambda^{-1}B^{-1} \in \Phi(X)$. Since $0 \in \rho(A) \cap \rho(B)$, $A, B \in \Phi(X)$ and $ind(A) = ind(B) = 0$. We can conclude that $A - \lambda B \in \Phi(X)$. Thus $\lambda \in \mathbb{K} \backslash \sigma_F(A,B)$. On the other

DOI: https://dx.doi.org/10.30504/JIMS.2024.446376.1163

hand, from (3.1), we have

$$\begin{aligned} ind(A-\lambda B) &= ind(-\lambda B(A^{-1}-\lambda^{-1}B^{-1})A) \\ &= ind(B)+ind(A)+ind(A^{-1}-\lambda^{-1}B^{-1}) \\ &= ind(A^{-1}-\lambda^{-1}B^{-1}). \end{aligned}$$

Conversely, let $0\neq\lambda\in\mathbb{K}\backslash\sigma_F(A,B)$, hence $(A-\lambda B)\in\Phi(X)$, then by (3.1), $B(A^{-1}-\lambda^{-1}B^{-1})A\in\Phi(X)$. Since $A,B\in\Phi(X)$, $A^{-1}-\lambda^{-1}B^{-1}\in\Phi(X)$, thus $\frac{1}{\lambda}\notin\sigma_F(A^{-1},B^{-1})$. □

From [9, Definition 2.3], we have the following:

Definition 3.6. *Let $A\in\mathcal{C}(X)$, $B\in\mathcal{L}(X)$ and $\varepsilon>0$. The pseudospectrum $\sigma_\varepsilon(A,B)$ of a operator pencil (A,B) of the form $A-\lambda B$ on X is defined by*

$$\sigma_\varepsilon(A,B)=\sigma(A,B)\cup\{\lambda\in\mathbb{K}:\|(A-\lambda B)^{-1}\|>\varepsilon^{-1}\}.$$

The pseudoresolvent $\rho_\varepsilon(A,B)$ of a operator pencil (A,B) of the form $A-\lambda B$ is defined by

$$\rho_\varepsilon(A,B)=\rho(A,B)\cap\{\lambda\in\mathbb{K}:\|(A-\lambda B)^{-1}\|\leq\varepsilon^{-1}\},$$

by convention $\|(A-\lambda B)^{-1}\|=\infty$, if$\lambda\in\sigma(A,B)$.

Now, we give a characterization of the essential spectrum of non-Archimedean operator pencils as follows.

Proposition 3.7. *Let X be a non-Archimedean Banach space over $\mathbb{Q}_p$, let $A,B\in\mathcal{L}(X)$. Then*

$$\bigcap_{K\in\mathcal{K}(X)}\sigma(A+K,B)=\{\lambda\in\mathbb{Q}_p:A-\lambda B\notin\Phi(X)\}\cup\{\lambda\in\mathbb{Q}_p:ind(A-\lambda B)\neq 0\}.$$

Proof. Let $\lambda\notin\{\lambda\in\mathbb{Q}_p:A-\lambda B\notin\Phi(X)\}\cup\{\lambda\in\mathbb{Q}_p:ind(A-\lambda B)\neq 0\}$. Then $A-\lambda B\in\Phi(X)$, and $ind(A-\lambda B)=0$. By Lemma 3.1, there is $K\in\mathcal{K}(X)$ such that $\lambda\in\rho(A+K,B)$. Thus $\lambda\notin\bigcap_{K\in\mathcal{K}(X)}\sigma(A+K,B)$. Hence

$$\bigcap_{K\in\mathcal{K}(X)}\sigma(A+K,B)\subseteq\{\lambda\in\mathbb{Q}_p:A-\lambda B\notin\Phi(X)\}\cup\{\lambda\in\mathbb{Q}_p:ind(A-\lambda B)\neq 0\}.$$

Let $\lambda\notin\bigcap_{K\in\mathcal{K}(X)}\sigma(A+K,B)$, then $A+K-\lambda B\in\Phi(X)$, and $ind(A+K-\lambda B)=0$. Hence $A-\lambda B=A-\lambda B+K-K$. By Theorem 2.9, $A-\lambda B\in\Phi(X)$ and $ind(A+K-\lambda B)=ind(A-\lambda B)=0$. Consequently,

$$\lambda\notin\{\lambda\in\mathbb{Q}_p:A-\lambda B\notin\Phi(X)\}\cup\{\lambda\in\mathbb{Q}_p:ind(A-\lambda B)\neq 0\}.$$

This completes the proof. □

From the definition of the pseudospectrum of operator pencils, we deduce the following theorem.

Theorem 3.8. *Let X be a non-Archimedean Banach space over $\mathbb{K}$. Let $A\in\mathcal{C}(X),B\in\mathcal{L}(X)$ and $\varepsilon>0$. Then*

$$\sigma_\varepsilon(A,B)=\sigma(A,B)\cup\{\lambda\in\mathbb{K}:\exists\, x\in D(A),\|(A-\lambda B)x\|<\varepsilon\|x\|\}.$$

DOI: https://dx.doi.org/10.30504/JIMS.2024.446376.1163

Proof. Let $\lambda \in \sigma_\varepsilon(A,B)$, then $\lambda \in \sigma(A,B)$ or $\|(A-\lambda B)^{-1}\| > \frac{1}{\varepsilon}$. If $\lambda \in \sigma_\varepsilon(A,B)$, and $\lambda \notin \sigma(A,B)$, then there exists $y \in X\backslash\{0\}$ such that

$$\frac{\|(A-\lambda B)^{-1}y\|}{\|y\|} > \frac{1}{\varepsilon}. \tag{3.2}$$

Set $x = (A-\lambda B)^{-1}y$ with $x \in D(A)$. By (3.2),

$$\frac{\|x\|}{\|(A-\lambda B)x\|} > \frac{1}{\varepsilon}.$$

Thus there exists $x \in D(A)$ such that $\|(A-\lambda B)x\| < \varepsilon\|x\|$. Conversely, let $\lambda \in \mathbb{K}$ such that there exists $x \in D(A)$ and

$$\|(A-\lambda B)x\| < \varepsilon\|x\| \tag{3.3}$$

or $\lambda \in \sigma(A,B)$. If $\lambda \notin \sigma(A,B)$ and put $y = (A-\lambda B)x$, then $x = (A-\lambda B)^{-1}y$. Hence by (3.3),

$$\|y\| < \varepsilon\|(A-\lambda B)^{-1}y\|.$$

Since $y \neq 0$, it follows that

$$\frac{1}{\varepsilon} < \|(A-\lambda B)^{-1}\|,$$

then $\lambda \in \sigma_\varepsilon(A,B)$. □

As the classical setting, we have the following theorem.

Theorem 3.9. *Let X be a non-Archimedean Banach space over $\mathbb{K}$ such that $\|X\| \subseteq |\mathbb{K}|$, and let $A \in \mathcal{C}(X)$, $B \in \mathcal{L}(X)$, and $\varepsilon > 0$. Then*

$$\sigma_\varepsilon(A,B) = \sigma(A,B) \cup \{\lambda \in \mathbb{K} : \exists\, x_n \in D(A),\ \|x_n\| = 1 \text{ and } \lim_{n\to\infty} \|(A-\lambda B)x_n\| < \varepsilon\}.$$

The next corollary is essential in the proof of Proposition 3.11.

Corollary 3.10. *For all $\lambda \in \sigma(A,B)$ and $\mu \in \mathbb{K}$, we have $\lambda + \mu \in \sigma(A+\mu B, B)$.*

Proof. If $\lambda + \mu \in \rho(A+\mu B, B)$, then $(A+\mu B-(\lambda+\mu)B)^{-1} \in \mathcal{L}(X)$, hence $(A-\lambda B)^{-1} \in \mathcal{L}(X)$ which is a contradiction. □

In the following proposition, we collect some properties of non-Archimedean pseudospectrum of operator pencils.

Proposition 3.11. *Let X be a non-Archimedean Banach space over a spherically complete field $\mathbb{K}$ such that $\|X\| \subseteq |\mathbb{K}|$. Let $A \in \mathcal{C}(X)$, $B \in \mathcal{L}(X)$, such that $\|B\| \leq 1$, and $\varepsilon, \delta > 0$ .Then*

(i) *$\sigma(A,B) + B(0,\varepsilon) \subseteq \sigma_\varepsilon(A,B)$, where $B(0,\varepsilon)$ is the open disk centered at zero with radius ε;*

(ii) *$\sigma_\varepsilon(A,B) + B(0,\delta) \subseteq \sigma_{\varepsilon+\delta}(A,B)$.*

Proof.

(i) Let $\lambda \in \sigma(A,B) + B(0,\varepsilon)$, then there is $\lambda_1 \in \sigma(A,B)$ and $\lambda_2 \in B(0,\varepsilon)$ such that $\lambda = \lambda_1 + \lambda_2$. Since $\lambda_1 \in \sigma(A,B)$, from Corollary 3.10, $\lambda_1 + \lambda_2 \in \sigma(A+\lambda_2 B, B)$. Also $|\lambda_2|\|B\| < \varepsilon$. Set $D = \lambda_2 B$. Then $D \in \mathcal{L}(X)$, $\|D\| < \varepsilon$ and $\lambda \in \sigma(A+D,B)$. By Theorem 2.17, $\lambda \in \sigma_\varepsilon(A,B)$.

DOI: https://dx.doi.org/10.30504/JIMS.2024.446376.1163

(ii) Let $\lambda \in \sigma_\varepsilon(A,B) + B(0,\delta)$, then there is $\lambda_1 \in \sigma_\varepsilon(A,B)$ and $\lambda_2 \in B(0,\delta)$ such that $\lambda = \lambda_1 + \lambda_2$. Since $\lambda_1 \in \sigma_\varepsilon(A,B)$, by Theorem 2.17, there is $C \in \mathcal{L}(X)$ such that $\|C\| < \varepsilon$ and $\lambda_1 \in \sigma(A+C,B)$. By Corollary 3.10, $\lambda = \lambda_1 + \lambda_2 \in \sigma(A + C + \lambda_2 B, B)$. Also, we have $C + \lambda_2 B \in \mathcal{L}(X)$ with

$$\|C + \lambda_2 B\| \leq \max\{\|C\|, |\lambda_2|\|B\|\} < \max\{\varepsilon, \delta\} < \varepsilon + \delta.$$

From Theorem 2.17, we conclude that $\lambda \in \sigma_{\varepsilon+\delta}(A,B)$. □

The next proposition gives a relationship between the spectrum of AB and the spectrum of BA.

Proposition 3.12. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, and let $A \in \mathcal{L}(X)$, then $1 \notin \sigma(AB)$ if and only if $1 \notin \sigma(BA)$.*

Proof. Let $1 \notin \sigma(AB)$, then $(I - AB)^{-1}$ is invertible, hence there is $C \in \mathcal{L}(X)$ such that

$$C(I - AB) = (I - AB)C = I.$$

Thus $C = I + CAB = I + ABC$, then $ABC = CAB$. Moreover,

$$\begin{aligned}
(I + BCA)(I - BA) &= I - BA + BCA - BCABA \\
&= I - BA + BC(I - AB)A \\
&= I - BA + BA \\
&= I,
\end{aligned}$$

and

$$\begin{aligned}
(I - BA)(I + BCA) &= I - BA + BCA - BABCA \\
&= I - BA + BCA - BCABA \text{ since } ABC = CAB \\
&= I - BA + BC(I - AB)A \\
&= I - BA + BA \\
&= I.
\end{aligned}$$

Hence $I + BCA$ is the inverse of $I - BA$. Consequently, $1 \notin \sigma(BA)$. Similarly, we obtain that if $1 \notin \sigma(BA)$, then $1 \notin \sigma(AB)$. □

We introduce the following definition.

Definition 3.13. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, such that $\|X\| \subseteq |\mathbb{K}|$. Let $A \in \mathcal{C}(X)$, $B, C \in \mathcal{L}(X)$ and $\varepsilon > 0$. The structured pseudospectrum $\sigma_\varepsilon(A,B,C)$ of A is defined by*

$$\sigma_\varepsilon(A,B,C) = \bigcup_{D \in \mathcal{L}(X) : \|D\| < \varepsilon} \sigma(A + CDB).$$

Remark 3.14. *Let $A \in \mathcal{C}(X)$, $B, C \in \mathcal{L}(X)$ and $\varepsilon > 0$. If $C = B = I$, then $\sigma_\varepsilon(A,I,I) = \sigma_\varepsilon(A)$ is the pseudospectrum of A.*

DOI: https://dx.doi.org/10.30504/JIMS.2024.446376.1163

The following theorem gives a characterization of the structured pseudospectrum of operator pencils on non-Archimedean Banach spaces.

Theorem 3.15. *Let X be a non-Archimedean Banach space over a spherically complete field $\mathbb{K}$ such that $\|X\| \subseteq |\mathbb{K}|$. Let $A \in \mathcal{C}(X)$, $B, C \in \mathcal{L}(X)$ such that $0 \in \rho(B) \cap \rho(C)$ and $\varepsilon > 0$. Then,*

$$\sigma_\varepsilon(A, B, C) = \sigma(A) \cup \{\lambda \in \mathbb{K} : \|B(A - \lambda I)^{-1}C\| > \frac{1}{\varepsilon}\}.$$

Proof. If $D = 0$, we have

$$\sigma(A) \subseteq \sigma_\varepsilon(A, B, C).$$

If $D \neq 0$, let $\lambda \notin \sigma(A)$. If $\|B(A - \lambda I)^{-1}C\| \leq \varepsilon^{-1}$. Then for all $D \in \mathcal{L}(X) : \|D\| < \varepsilon$. Hence $\|DB(A - \lambda I)^{-1}C\| < 1$. Therefore, $I - DB(A - \lambda I)^{-1}C$ is invertible. By Proposition 3.12, for all $D \in \mathcal{L}(X) : \|D\| < \varepsilon,\ 1 \notin \sigma(DB(A - \lambda I)^{-1}C)$ if and only if $1 \notin \sigma(CDB(A - \lambda I)^{-1})$. Thus

$$A + CDB - \lambda I = (I + CDB(A - \lambda I)^{-1})(A - \lambda I).$$

Consequently,

$$\lambda \notin \bigcup_{D\in\mathcal{L}(X):\|D\|<\varepsilon} \sigma(A + CDB).$$

For the converse inclusion, if $\lambda \notin \sigma(A)$, then $\|B(A - \lambda I)^{-1}C\| > \varepsilon^{-1}$. Hence

$$\sup_{x\in X\setminus\{0\}} \frac{\|B(A - \lambda I)^{-1}Cx\|}{\|x\|} > \frac{1}{\varepsilon}.$$

Thus there exists $x \in X\backslash\{0\}$ such that

$$\|B(A - \lambda I)^{-1}Cx\| > \frac{\|x\|}{\varepsilon}. \tag{3.4}$$

Set $y = B(A - \lambda I)^{-1}Cx$, thus $C^{-1}(A - \lambda I)B^{-1}y = x$. From (3.4),

$$\|C^{-1}(A - \lambda I)B^{-1}y\| < \varepsilon\|y\|. \tag{3.5}$$

Since $\|X\| \subseteq |\mathbb{K}|$, there is $c \in \mathbb{K}\backslash\{0\}$ such that $\|y\| = |c|$, set $z = c^{-1}y$ hence $\|z\| = 1$. From (3.5), $\|C^{-1}(A-\lambda I)B^{-1}z\| < \varepsilon$. By Theorem 2.12, there is $\phi \in X^*$ such that $\phi(z) = 1$ and $\|\phi\| = \|z\|^{-1} = 1$. Put for all $x \in X, Dx = \phi(x)C^{-1}(\lambda I - A)B^{-1}z$. Hence

$$\begin{aligned}\|Dx\| &= |\phi(x)|\|C^{-1}(A - \lambda I)B^{-1}z\| \\ &\leq \|\phi\|\|x\|\|C^{-1}(A - \lambda I)B^{-1}z\| \\ &< \varepsilon\|x\|.\end{aligned}$$

So $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$. Moreover for $z \neq 0$, we have $(A + CDB - \lambda I)z = 0$, thus $A + CDB - \lambda I$ is not injective, then $A + CDB - \lambda I$ is not invertible. Using Definition 3.13, $\lambda \in \sigma_\varepsilon(A, B, C)$. □

Now, we collect some properties of non-Archimedean structured pseudospectrum of operators pencils.

Theorem 3.16. *Let X be a non-Archimedean Banach space over a spherically complete field $\mathbb{K}$ such that $\|X\| \subseteq |\mathbb{K}|$. Let $A \in \mathcal{C}(X)$, $B, C \in \mathcal{L}(X)$ such that $0 \in \rho(B) \cap \rho(C)$ and $\varepsilon > 0$. Then,*

DOI: https://dx.doi.org/10.30504/JIMS.2024.446376.1163

(i) *For all* $\varepsilon_1, \varepsilon_2$ *such that* $\varepsilon_1 \leq \varepsilon_2$, *we have* $\sigma_{\varepsilon_1}(A,B,C) \subseteq \sigma_{\varepsilon_2}(A,B,C)$;

(ii) $\sigma(A) = \bigcap_{\varepsilon>0} \sigma_\varepsilon(A,B,C)$.

Proof.

(i) Let $\lambda \in \sigma_{\varepsilon_1}(A,B,C)$, then by Theorem 3.15, $\|B(A-\lambda I)^{-1}C\| > \varepsilon_1^{-1} \geq \varepsilon_2^{-1}$. Thus $\lambda \in \sigma_{\varepsilon_2}(A,B,C)$.

(ii) Since for all $\varepsilon > 0$, $\sigma(A) \subseteq \sigma_\varepsilon(A,B,C)$, we have $\sigma(A) \subseteq \bigcap_{\varepsilon>0} \sigma_\varepsilon(A,B,C)$.

Conversely, if $\lambda \in \bigcap_{\varepsilon>0} \sigma_\varepsilon(A,B,C)$, then for all $\varepsilon > 0$, $\lambda \in \sigma_\varepsilon(A,B,C)$. If $\lambda \notin \sigma(A)$, then $\lambda \in \{\lambda \in \mathbb{K} : \|B(A-\lambda I)^{-1}C\| > \varepsilon^{-1}\}$, taking limits as $\varepsilon \to 0^+$, we get $\|B(A-\lambda I)^{-1}C\| = \infty$. Thus $\lambda \in \sigma(A)$. □

Theorem 3.17. *Let X be a non-Archimedean Banach space over a spherically complete field $\mathbb{K}$ such that $\|X\| \subseteq |\mathbb{K}|$. Let $A \in \mathcal{C}(X)$, $B, C \in \mathcal{L}(X)$ such that $0 \in \rho(B) \cap \rho(C)$, $B(D(A)) = D(A)$ and $\varepsilon > 0$. Then,*

$$\sigma_\varepsilon(A,B,C) = \sigma(A) \cup \{\lambda \in \mathbb{K} : \exists\, x \in D(A), \|x\| = 1,\ \|C^{-1}(A-\lambda I)B^{-1}x\| < \varepsilon\}.$$

Proof. From Theorem 3.15,

$$\sigma_\varepsilon(A,B,C) = \sigma(A) \cup \{\lambda \in \mathbb{K} : \|B(A-\lambda I)^{-1}C\| > \frac{1}{\varepsilon}\}.$$

Let $\lambda \in \sigma_\varepsilon(A,B,C)\backslash\sigma(A)$, then $\|B(A-\lambda I)^{-1}C\| > \varepsilon^{-1}$. Thus there exists $x \in X\backslash\{0\}$ such that

$$\|B(A-\lambda I)^{-1}Cx\| > \frac{\|x\|}{\varepsilon}. \tag{3.6}$$

Set $y = B(A-\lambda I)^{-1}Cx$ with $y \in D(A)$, thus $C^{-1}(A-\lambda I)B^{-1}y = x$. From (3.6),

$$\|C^{-1}(A-\lambda I)B^{-1}y\| < \varepsilon\|y\|. \tag{3.7}$$

Since $\|X\| \subseteq |\mathbb{K}|$, there is $c \in \mathbb{K}\backslash\{0\}$ such that $\|y\| = |c|$, set $z = c^{-1}y$, hence $\|z\| = 1$. By (3.7), $\|C^{-1}(A-\lambda I)B^{-1}y\| < \varepsilon$. Conversely, assume that there is $z \in D(A)$ such that $\|z\| = 1$ and $\|C^{-1}(A-\lambda I)B^{-1}y\| < \varepsilon$. By Theorem 2.12, there is $\phi \in X^*$ such that $\phi(z) = 1$ and $\|\phi\| = \|z\|^{-1} = 1$. Set for any $x \in X$, $Dx = \phi(x)C^{-1}(\lambda I - A)B^{-1}z$. Hence for each $x \in X$,

$$\begin{aligned} \|Dx\| &= |\phi(x)|\|C^{-1}(A-\lambda I)B^{-1}z\| \\ &\leq \|\phi\|\|x\|\|C^{-1}(A-\lambda I)B^{-1}z\| \\ &< \varepsilon\|x\|. \end{aligned}$$

Then $D \in \mathcal{L}(X) : \|D\| < \varepsilon$. Moreover for $z \neq 0$, we have $(A + CDB - \lambda I)z = 0$, thus $A + CDB - \lambda I$ is not injective, then $A + CDB - \lambda I$ is not invertible. By Definition 3.13, $\lambda \in \sigma_\varepsilon(A,B,C)$. □

We have the following definition.

Definition 3.18. *Let X be a non-Archimedean Banach space over $\mathbb{Q}_p$ such that $\|X\| \subseteq |\mathbb{Q}_p|$. Let $A \in \mathcal{C}(X)$, $B, C \in \mathcal{L}(X)$ and $\varepsilon > 0$. The structured Fredholm pseudospectrum $\sigma_{F,\varepsilon}(A,B,C)$ of A is*

DOI: https://dx.doi.org/10.30504/JIMS.2024.446376.1163

given by

$$\sigma_{F,\varepsilon}(A,B,C)=\bigcup_{D\in\mathcal{L}(X):\|D\|<\varepsilon}\sigma_F(A+CDB).$$

Remark 3.19. *Let X be a non-Archimedean Banach space over $\mathbb{K}$. Let $A\in\mathcal{C}(X)$, $B,C,D\in\mathcal{L}(X)$. Then,*

$$(\lambda I-A)(\lambda I-CDB)=ACDB+\lambda(\lambda I-A-CDB) \tag{3.8}$$

and

$$(\lambda I-CDB)(\lambda I-A)=CDBA+\lambda(\lambda I-A-CDB). \tag{3.9}$$

We obtain the following theorem.

Theorem 3.20. *Let X be a non-Archimedean Banach space over $\mathbb{Q}_p$ such that $\|X\|\subseteq|\mathbb{Q}_p|$. Let $A\in\mathcal{C}(X)$, $B,C\in\mathcal{L}(X)$ and $\varepsilon>0$. If for all $D\in\mathcal{L}(X)$ such that $\|D\|<\varepsilon$, $ACDB$ is a Fredholm operator, then*

$$\sigma_{F,\varepsilon}(A,B,C)\backslash\{0\}\subset\Big[\sigma_F(A)\cup\bigcup_{D\in\mathcal{L}(X):\|D\|<\varepsilon}\sigma_F(CDB)\Big]\backslash\{0\}.$$

Moreover, if for all $D\in\mathcal{L}(X):\|D\|<\varepsilon$, $CDBA$ and $ACDB$ are Fredholm operators, then

$$\sigma_{F,\varepsilon}(A,B,C)\backslash\{0\}=\Big[\sigma_F(A)\cup\bigcup_{D\in\mathcal{L}(X):\|D\|<\varepsilon}\sigma_F(CDB)\Big]\backslash\{0\}.$$

Proof. If $\lambda\notin\Big[\sigma_F(A)\cup\bigcup_{D\in\mathcal{L}(X):\|D\|<\varepsilon}\sigma_F(CDB)\Big]\backslash\{0\}$ or $\lambda=0$. If $\lambda\neq0$, then $\lambda I-A\in\Phi(X)$ and for all $D\in\mathcal{L}(X)$ such that $\|D\|<\varepsilon$, $\lambda I-CDB\in\Phi(X)$. From Theorem 2.16, for all $D\in\mathcal{L}(X)$ such that $\|D\|<\varepsilon$, $(\lambda I-A)(\lambda I-CDB)\in\Phi(X)$. Since $ACDB$ is Fredholm, using (3.8), we get for all $D\in\mathcal{L}(X):\ \|D\|<\varepsilon$, $\lambda I-A-CDB\in\Phi(X)$. Consequently, $\lambda\notin\sigma_{F,\varepsilon}(A,B,C)\backslash\{0\}$. For the converse inclusion, let $\lambda\notin\sigma_{F,\varepsilon}(A,B,C)\backslash\{0\}$, then for all $D\in\mathcal{L}(X)$ such that $\|D\|<\varepsilon$, $\lambda\notin\sigma_F(A+CDB)$ or $\lambda=0$. If $\lambda\neq0$, then for all $D\in\mathcal{L}(X)$ such that $\|D\|<\varepsilon$, $\lambda I-A-CDB\in\Phi(X)$. Since for all $D\in\mathcal{L}(X)$ such that $\|D\|<\varepsilon$, $ACDB$ and $CDBA$ are Fredholm. Thus for each $\|D\|<\varepsilon$, $(\lambda I-A)(\lambda I-CDB)\in\Phi(X)$ and $(\lambda I-CDB)(\lambda I-A)\in\Phi(X)$. By Lemma 3.3, $\lambda I-A\in\Phi(X)$ and for all $D\in\mathcal{L}(X)$ such that $\|D\|<\varepsilon$, $\lambda-CDB\in\Phi(X)$. Hence

$$\lambda\notin\Big[\sigma_F(A)\cup\bigcup_{D\in\mathcal{L}(X):\|D\|<\varepsilon}\sigma_F(CDB)\Big]\backslash\{0\}.$$

□

Definition 3.21. *Let X be a non-Archimedean Banach space over $\mathbb{Q}_p$, and let $A\in\mathcal{C}(X)$, $B,C\in\mathcal{L}(X)$ and $\varepsilon>0$. The structured essential pseudospectrum $\sigma_{e,\varepsilon}(A,B,C)$ of the linear operator A is given by*

$$\sigma_{e,\varepsilon}(A,B,C)=\bigcup_{D\in\mathcal{L}(X):\|D\|<\varepsilon}\sigma_e(A+CDB)$$

where $\sigma_e(M)=\{\lambda\in\mathbb{Q}_p: M-\lambda I$ is not Fredholm of index $0\}$ for $M\in\mathcal{L}(X)$.

DOI: https://dx.doi.org/10.30504/JIMS.2024.446376.1163

Now, we characterize the structured essential pseudospectrum of non-Archimedean bounded linear operator pencils as follows.

Theorem 3.22. *Let X be a non-Archimedean Banach space over $\mathbb{Q}_p$ such that $\|X\| \subseteq |\mathbb{Q}_p|$. Let $A \in \mathcal{L}(X)$, $B, C, D \in \mathcal{L}(X)$, $\lambda \in \mathbb{Q}_p$ and $\varepsilon > 0$ such that A^*, B^*, C^*, D^* exist and $N\Big((A + CDB - \lambda I)^*\Big) = R(A + CDB - \lambda I)^{\perp}$. Then $\lambda \notin \sigma_{e,\varepsilon}(A, B, C)$, if and only if $\lambda \notin \bigcap\limits_{K \in \mathcal{K}(X)} \sigma_{\varepsilon}(A + K, B, C)$.*

Proof. Let $\lambda \notin \sigma_{e,\varepsilon}(A, B, C)$, then for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, we have $A+CDB-\lambda I \in \Phi(X)$ and $ind(A+CDB-\lambda I) = 0$. Put $\alpha(A+CDB-\lambda I) = \beta(A+CDB-\lambda I) = n$. Let $\{x_1, \cdots, x_n\}$ being the basis for $N(A+CDB-\lambda I)$ and $\{y_1^*, \cdots, y_n^*\}$ being the basis for $R(A+CDB-\lambda I)^{\perp}$. By Lemma 2.10, there are functionals $x_1^*, \cdots, x_n^*$ in X^* $\big($ X^* is the dual space of $X\big)$ and elements $y_1, \cdots, y_n$ in X such that

$$x_j^*(x_k) = \delta_{j,k} \text{ and } y_j^*(y_k) = \delta_{j,k},\ 1 \leq j,\ k \leq n,$$

where $\delta_{j,k} = 0$ if $j \neq k$ and $\delta_{j,k} = 1$ if $j = k$. Consider the operator K defined on X by

$$\begin{aligned} K: \quad X &\to \quad X \\ x &\mapsto \quad \sum_{i=1}^{n} x_i^*(x) y_i. \end{aligned}$$

It is easy to see that K is linear operator and $D(K) = X$. In fact, for all $x \in X$,

$$\begin{aligned} \|Kx\| &= \|\sum_{i=1}^{n} x_i^*(x) y_i\| \\ &\leq \max_{1 \leq i \leq n} \|x_i^*(x) y_i\| \\ &\leq \max_{1 \leq i \leq n} (\|x_i^*\| \|y_i\|) \|x\|. \end{aligned}$$

Moreover, $R(K)$ is contained in a finite-dimensional subspace of X. So, K is a finite rank operator, then K is completely continuous. Since $\mathbb{K} = \mathbb{Q}_p$, from Remark 2.8, K is a compact operator. We show that for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, we have

$$N(A + CDB - \lambda I) \cap N(K) = \{0\} \tag{3.10}$$

and

$$R(A + CDB - \lambda I) \cap R(K) = \{0\}. \tag{3.11}$$

Let $x \in N(A+CDB-\lambda I) \cap N(K)$, then $x \in N(A+CDB-\lambda I)$ and $x \in N(K)$. If $x \in N(A+CDB-\lambda I)$, then

$$x = \sum_{i=1}^{n} \alpha_i x_i \text{ with } \alpha_1, \cdots, \alpha_n \in \mathbb{Q}_p.$$

DOI: https://dx.doi.org/10.30504/JIMS.2024.446376.1163

Then for all $1 \leq j \leq n$, $x_j^*(x) = \sum_{i=1}^{n} \alpha_i \delta_{i,j} = \alpha_j$. On the other hand, if $x \in N(K)$, then $Kx = 0$, so

$$\sum_{j=1}^{n} x_j^*(x) y_j = 0.$$

Therefore, we have for all $1 \leq j \leq n$, $x_j^*(x) = 0$. Hence $x = 0$. Consequently, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$,

$$N(A + CDB - \lambda I) \cap N(K) = \{0\}.$$

Let $y \in R(A + CDB - \lambda I) \cap R(K)$, then $y \in R(A + CDB - \lambda I)$ and $y \in R(K)$. Let $y \in R(K)$, we have

$$y = \sum_{i=1}^{n} \alpha_i y_i \text{ with } \alpha_1, \cdots, \alpha_n \in \mathbb{Q}_p.$$

Then for all $1 \leq j \leq n$, $y_j^*(y) = \sum_{i=1}^{n} \alpha_i \delta_{i,j} = \alpha_j$. On the other hand, if $y \in R(A + CDB - \lambda I)$, hence for all $1 \leq j \leq n$, $y_j^*(y) = 0$. Thus $y = 0$. Therefore,

$$R(A + CDB - \lambda I) \cap R(K) = \{0\}.$$

On the other hand, K is a compact operator. By Theorem 2.9, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, $A + CDB + K - \lambda I \in \Phi(X)$ and $ind(A + CDB + K - \lambda I) = 0$. Thus for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$,

$$\alpha(A + CDB + K - \lambda I) = \beta(A + CDB + K - \lambda I). \tag{3.12}$$

If $x \in N(A+CDB+K-\lambda I)$, then $(A+CDB-\lambda I)x = -Kx$ in $R(A+CDB-\lambda I)\cap R(K)$. It follows from (3.11) that $(A+CDB-\lambda I)x = -Kx = 0$, hence $x \in N(A+CDB-\lambda I)\cap N(K)$ and from (3.10), $x = 0$. Thus $\alpha(A + K + CDB - \lambda I) = 0$, it follows from (3.12) that $R(A + CDB + K - \lambda I) = X$. Consequently, $A + K + CDB - \lambda I$ is invertible and by Definition 3.13, $\lambda \notin \bigcap_{K \in \mathcal{K}(X)} \sigma_\varepsilon(A + K, B, C)$.

Let $\lambda \notin \bigcap_{K \in \mathcal{K}(X)} \sigma_\varepsilon(A + K, B, C)$, then there is $K \in \mathcal{K}(X)$ such that $\lambda \in \rho_\varepsilon(A + K, B, C)$, from Definition 3.13, there is $K \in \mathcal{K}(X)$ such that for all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, we have

$$A + CDB + K - \lambda I \in \Phi(X)$$

and

$$ind(A + CDB + K - \lambda I) = 0.$$

By Theorem 2.9, for each $D \in \mathcal{L}(X)$ satisfying $\|D\| < \varepsilon$, we have

$$A + CDB - \lambda I \in \Phi(X)$$

and

$$ind(A + CDB - \lambda I) = ind(A + CDB + K - \lambda I) = 0.$$

Consequently, $\lambda \notin \sigma_{e,\varepsilon}(A, B, C)$. □

DOI: https://dx.doi.org/10.30504/JIMS.2024.446376.1163

We finish with the following example.

Example 3.23. *Let X be a free Banach space over $\mathbb{Q}_p$ such that $\|X\| \subseteq |\mathbb{Q}_p|$. Let $A, B, C \in \mathcal{L}(X)$ be diagonal operators such that $0 \in \rho(B) \cap \rho(C)$ and for all $i \in \mathbb{N}$, $Ae_i = a_i e_i, Be_i = b_i e_i$ and $Ce_i = c_i e_i$ with $(a_i)_{i\in\mathbb{N}}, (b_i)_{i\in\mathbb{N}}, (c_i)_{i\in\mathbb{N}} \subset \mathbb{Q}_p : \sup\limits_{i\in\mathbb{N}} |a_i|_p, \sup\limits_{i\in\mathbb{N}} |b_i|_p, \sup\limits_{i\in\mathbb{N}} |c_i|_p$ are finite. From* [6, Proposition 3.55],

$$\sigma(A) = \{\lambda \in \mathbb{Q}_p : \inf_{i\in\mathbb{N}} |a_i - \lambda| = 0\} = \overline{\{a_i : \ i \in \mathbb{N}\}},$$

and for all $\lambda \in \rho(A)$, we have

$$\begin{aligned} \|B(A-\lambda I)^{-1}C\| &= \sup_{i\in\mathbb{N}} \frac{\|B(A-\lambda I)^{-1}Ce_i\|}{\|e_i\|} \\ &= \sup_{i\in\mathbb{N}} \left|\frac{b_i c_i}{a_i - \lambda}\right|. \end{aligned}$$

Consequently,

$$\sigma_\varepsilon(A, B, C) = \overline{\{a_i : \ i \in \mathbb{N}\}} \cup \left\{\lambda \in \mathbb{Q}_p : \sup_{i\in\mathbb{N}} \left|\frac{b_i c_i}{a_i - \lambda}\right| > \frac{1}{\varepsilon}\right\}.$$

For more examples of non-Archimedean structured pseudospectrum of matrices, we refer the readers to [12].

References

[1] F. Abdmouleh, A. Ammar and A. Jeribi, Stability of the S-Essential Spectra on a Banach Space, *Math. Slovaca* **63** (2013), no. 2, 299–320.

[2] A. Ammar, A. Bouchekoua and A. Jeribi, Pseudospectra in a Non-Archimedean Banach Space and Essential Pseudospectra in E_ω, *Filomat* **33** (2019), no. 12, 3961–3976.

[3] J. Araujo, C. Perez-Garcia and S. Vega, Preservation of the index of p-adic linear operators under compact perturbations, *Compositio Math.* **118** (1999), no. 3, 291–303.

[4] A. Blali, A. El Amrani and J. Ettayb, Some spectral sets of linear operator pencils on non-Archimedean Banach spaces, *Bull. Transilv. Univ. Braşov Ser. III. Math. Comput. Sci.* **2(64)** (2022), no. 1, 41–56.

[5] A. Blali, A. El Amrani and J. Ettayb, A note on Pencil of bounded linear operators on non-Archimedean Banach spaces, *Methods Funct. Anal. Topology* **28** (2022), no. 2, 105–109.

[6] T. Diagana and F. Ramaroson, Non-archimedean Operators Theory, Springer, 2016.

[7] E. B. Davies, Linear Operators and Their Spectra, Cambridge University Press, New York, 2007.

[8] A El Amrani, J Ettayb and A Blali, Pseudospectrum and condition pseudospectrum of non-archimedean matrices, *J. Prime Res. Math.* **18** (2022), no. 1, 75–82.

[9] A. El Amrani, A. Blali and J. Ettayb, On Pencil of Bounded Linear Operators on Non-archimedean Banach Spaces, *Bol. Soc. Paran. Mat.* **42** (2024), 1–10.

[10] J. Ettayb, Pseudospectrum and essential pseudospectrum of bounded linear operator pencils on non-Archimedean Banach spaces, Bol. Soc. Paran. Mat, to appear.

[11] J. Ettayb, Pseudospectrum of non-Archimedean matrix pencils, Bulletin of the Transilvania University of Braşov Series III: Mathematics and Computer Science, in press.

[12] J. Ettayb, Structured pseudospectrum and structured condition pseudospectrum of non-archimedean matrices, arXiv preprint arXiv:2211.10365, 2022.

[13] S. N. Krishnamachari, Linear Operators between Nonarchimedean Banach Spaces, Dissertations, Western Michigan University, Ann Arbor, 1973.

DOI: https://dx.doi.org/10.30504/JIMS.2024.446376.1163

[14] H. R. Henriquez, H. G. Samuel Navarro and J. Aguayo, Closed linear operators between nonarchimedean Banach spaces, *Indag. Math. (N.S.)* **16** (2005), no. 2, 201–214.

[15] A. Jeribi, Linear operators and their essential pseudospectra, Apple Academic Press, 2018.

[16] A. F. Monna, Analyse non-archimédienne, Springer, Berlin, 1970.

[17] C. Perez-Garcia and S. Vega, Perturbation theory of p-adic Fredholm and semi-Fredholm operators, *Indag. Math. (N.S.)* **15** (2004), no. 1, 115–128.

[18] A. C. M. van Rooij, Non-Archimedean functional analysis, Monographs and Textbooks in Pure and Applied Math. 51. Marcel Dekker, Inc., New York, 1978.

[19] L. N. Trefethen and M. Embree, Spectra and Pseudospectra, The behavior of nonnormal matrices and operators, Princeton University Press, Princeton, 2005.

Jawad Ettayb

Department of Mathematics, Sidi Mohamed Ben Abdellah University, Fez, Morocco.

Email: jawad.ettayb@usmba.ac.ma

DOI: https://dx.doi.org/10.30504/JIMS.2024.446376.1163