Journal of the Iranian Mathematical Society
ISSN (on-line): 2717-1612
J. Iran. Math. Soc. 4 (2023), no. 2, 179-188

ON THE WATCHING NUMBER OF GRAPHS

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Abstract. Let $G = (V, E)$ be a simple and undirected graph. A watcher ω_i of G is a couple of $\omega_i = (v_i, Z_i)$, where $v_i \in V$ and Z_i is a subset of the closed neighborhood of v_i. If a vertex $v \in Z_i$, we say that v is covered by ω_i. A set $W = \{\omega_1, \omega_2, \dots, \omega_k\}$, of watchers, is a watching system for G if the sets $L_W(v) = \{\omega_i \ : \ v \in Z_i \ , \ 1 \leq i \leq k\}$ are non-empty and distinct, for every $v \in V$. In this paper, we study the watching systems of some graphs and consider the watching number of Mycielski's construction of some graphs.

1. Introduction

In this paper, all graphs are assumed to be finite, simple and undirected. We will often use the notation $G = (V, E)$ to denote a graph with non-empty vertex set $V = V(G)$ and edge set $E = E(G)$. An edge of G with end vertices u and v is denoted by uv. The *order* of a graph is the number of vertices in the graph and the *size* of a graph is the number of edges. For every vertex $x \in V(G)$, the *open neighborhood* of the vertex x is denoted by $N_G(x)$ and defined as $N_G(x) = \{y \in V(G) \ : \ xy \in E(G)\}$. Also, the *closed neighborhood* of the vertex $x \in V(G)$, $N_G[x]$, is $N_G[x] = N_G(x) \cup \{x\}$. The *degree* of a vertex $x \in V(G)$ is $\deg_G(x) = \big|N_G(x)\big|$. The *maximum degree* of a graph G is denoted by $\Delta(G)$. A vertex $x \in V(G)$ is called a *universal vertex*, if $N_G[x] = V(G)$. A *complete bipartite graph* is a special kind of bipartite graph, where every vertex of the first part is connected to every vertex of the second part. The complete bipartite graph is denoted by $K_{r,s}$.

Communicated by Alireza Abdollahi

MSC(2020): Primary: 05C76.

Keywords: Identifying code, watching system, Mycielski's construction.

Received: 5 March 2023, Accepted: 10 July 2023.

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DOI: https://dx.doi.org/10.30504/JIMS.2023.388523.1097

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A subset S of the vertices of a graph G is a *dominating set* of G, if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The *domination number* of a graph G, denoted by $\gamma(G)$, is the cardinality of a minimum dominating set of G. A set C of vertices G is an *identifying code* of G, if for every two vertices x and y the sets $N_G[x] \cap C$ and $N_G[y] \cap C$ are non-empty and different. The smallest size of an *identifying code* of a graph G, is called the *identifying code number* of G and is denoted by $\gamma^{ID}(G)$. For more details, we refer the reader to [5,6,9,12].

For graph $G = (V, E)$, let $V'(G) = \{v'_1, v'_2, \ldots, v'_n\}$ be a copy of $V(G) = \{v_1, v_2, \ldots, v_n\}$ and w be a new vertex. *Mycielski's construction* of graph G, denoted by $\mu(G)$ and it is a graph with the vertex set $V(\mu(G)) = V(G) \cup V'(G) \cup \{w\}$ and the edge set $E(\mu(G)) = E(G) \cup \{v_i v'_j \ : \ v_i v_j \in E(G),\ 1 \le i, j \le n\} \cup \{wv'_i \ : \ 1 \le i \le n\}$. The Mycielski's construction of a graph G was introduced by J. Mycielski to construct triangle-free graphs with the arbitrarily large chromatic number [8]. In recent years, there have been results reported on the Mycielski graph related to several domination parameters [4,7]. Watching systems were introduced in [2], are generalization of identifying codes. A *watcher* ω of a graph G is a couple of $\omega = (v_i, Z_i)$, where $v_i \in V(G)$ and $Z_i \subseteq N_G[v_i]$. We will say that ω is located at v_i. A *watching system* for a graph G is a finite set $W = \{\omega_1, \omega_2, \ldots, \omega_k\}$, such that for every $v \in V(G)$, $L_W(v) = \{\omega_i \ : \ v \in Z_i\ ,\ 1 \le i \le k\}$ are non-empty and distinct. The *watching number* of a graph G is denoted by $\omega(G)$ is the minimum size of watching systems of G. Auger et al. in [3], gave an upper bound on $\omega(G)$ for connected graphs of order n and characterized the trees attaining this bound. In 2014, Maimani et al. in [10], studied the watching systems of triangular graphs. They proved that watching number of triangular graph $T(n)$ is equal to $\lceil \frac{2n}{3} \rceil$. In 2017, Roozbayani et al. in [11], studied identifying codes and watching systems in Kneser graphs.

In this paper, we study the watching number of some graphs. In Section 2, we give two bounds for $\omega(G)$, which are sharp. We show that if G is a connected graph, $\Delta(G) = n-2$ and $N_G(a) = N_G(b) = G \setminus \{a, b\}$, then $\omega(G) = \lceil log_2(n+1) \rceil$. In Section 3, we give two bounds for $\omega(\mu(G))$, which are sharp. We prove if G has a universal vertex, then $\omega(\mu(G)) = \lceil log_2(n+1) \rceil + 2$. Finally, we give an upper bound for $\omega(\mu(C_n))$.

2. Some results on watching systems

In this Section, we give some results about watching systems.

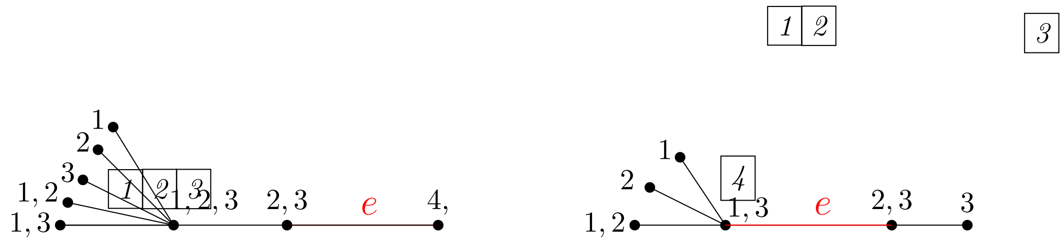
Theorem 2.1. [2] *Let G be a graph of order n. Then*

i) *If G is twin free graph, then $\gamma(G) \le \omega(G) \le \gamma^{ID}(G)$,*

ii) $\lceil log_2(n+1) \rceil \le \omega(G) \le \gamma(G) \lceil log_2(\Delta(G)+2) \rceil$.

Example 2.2. *For graphs G and H which are shown in Figure 1, we have $\omega(G) = 4$ and $\omega(H) = 3$.*

DOI: https://dx.doi.org/10.30504/JIMS.2023.388523.1097



G H

FIGURE 1

Theorem 2.3. *Let G be a graph of order n with $\Delta(G) = n - 1$. Then $\omega(G) = \lceil log_2(n+1)\rceil$.*

Proof. By Theorem 2.1, (ii), $\lceil log_2(n+1)\rceil \leq \omega(G)$. Since G has a universal vertex, $\gamma(G) = 1$. By Theorem 2.1, (ii), we have $\omega(G) \leq \lceil log_2(n-1+2)\rceil = \lceil log_2(n+1)\rceil$. So $\omega(G) = \lceil log_2(n+1)\rceil$. $\square$

Theorem 2.4. *Let G be a connected graph and $e \in E(G)$ such that $G \setminus \{e\} = G_1 \cup G_2$. Then $\omega(G_1) + \omega(G_2) - 2 \leq \omega(G) \leq \omega(G_1) + \omega(G_2)$.*

Proof. Let a and b be two end vertices of e, $a \in V(G_1)$ and $b \in V(G_2)$. Let W_i be a watching system for G_i with minimum cardinality, $i \in \{1, 2\}$ and $W_{12} = W_1 \cup\ W_2$. We claim that W_{12} is a watching system of G. For every $x \in V(G_1) \setminus \{a\}$ and $y \in V(G_2) \setminus \{b\}$, $L_{W_{12}}(x) = L_{W_1}(x)$ and $L_{W_{12}}(y) = L_{W_2}(y)$. Also, $L_{W_{12}}(a) \subseteq L_{W_1}(a) \cup \{(b, Z_b)\}$ and $L_{W_{12}}(b) \subseteq L_{W_2}(b) \cup \{(a, Z_a)\}$, where $Z_a \subseteq N_G[a]$ and $Z_b \subset N_G[b]$. Since W_1 and W_2 are watching systems for G_1 and G_2, respectively, so $L_{W_{12}}(v_i) \neq \emptyset$ and $L_{W_{12}}(v_i) \neq L_{W_{12}}(v_j)$ for every v_i and v_j in $V(G)$. Hence, W_{12} is a watching system for G and so $\omega(G) \leq |W_{12}| = |W_1| + |W_2| = \omega(G_1) + \omega(G_2)$.

Let W be a watching system for G with minimum cardinality. Then we have the following cases:

Case 1: Let $(a, Z_a) \notin W$ and $(b, Z_b) \notin W$, where $Z_a \subseteq N_G[a]$ and $Z_b \subseteq N_G[b]$. Also, let $W_1 = \{(v, Z_v) \in W\ :\ v \in V(G_1)\}$ and $W_2 = \{(u, Z_u) \in W\ :\ u \in V(G_2)\}$. Then W_1 and W_2 are watching systems for G_1 and G_2, respectively, $W_1 \cup\ W_2 = W$ and $W_1 \cap\ W_2 = \emptyset$. Thus $\omega(G_1) + \omega(G_2) \leq |W_1| + |W_2| = |W| = \omega(G)$. Hence $\omega(G_1) + \omega(G_2) - 2 \leq \omega(G)$.

Case 2: Let $(a, Z_a) \in W$ and $(b, Z_b) \notin W$, where $Z_a \subseteq N_G[a]$ and $Z_b \subseteq N_G[b]$. Then $W_1 = \{(v, Z_v \cap V(G_1)\ :\ v \in V(G_1),\ (v, Z_v) \in W\}$ and $W_2 = \{(u, Z_u) \in W\ :\ u \in V(G_2)\} \cup \{(b, \{b\})\}$ are watching systems for G_1 and G_2, respectively.

So $\omega(G_1) + \omega(G_2) \leq |W_1| + |W_2| = |W| + 1 = \omega(G) + 1$. Hence $\omega(G_1) + \omega(G_2) - 2 \leq \omega(G)$.

Case 3: Let $(a, Z_a) \in W$ and $(b, Z_b) \in W$.

If $(b, Z_b) \in L_W(a)$ and $L_W(a) \setminus \{(b, Z_b)\} = L_W(x)$ for some $x \in V(G_1)$, then

$$W_1 = \{(v, Z_v \cap V(G_1))\ :\ v \in V(G_1),\ (v, Z_v) \in W\} \cup \{(a, \{a\})\}$$

DOI: https://dx.doi.org/10.30504/JIMS.2023.388523.1097

is a watching system for G_1.
If $(b, Z_b) \in L_W(a)$ and $L_W(a) \setminus \{(b, Z_b)\} \neq L_W(x)$ for every $x \in V(G_1) \setminus \{a\}$, then

$$W_1 = \{(v, Z_v \cap V(G_1)) \ : \ v \in V(G_1),\ (v, Z_v) \in W\}$$

is a watching system for G_1.
If $(b, Z_b) \notin L_W(a)$, then

$$W_1 = \{(v, Z_v \cap V(G_1)) \ : \ v \in V(G_1),\ (v, Z_v) \in W\}$$

is a watching system for G_1.
Similarly, If $(a, Z_a) \in L_W(b)$ and $L_W(b) \setminus \{(a, Z_a)\} = L_W(x)$ for some $x \in V(G_2)$, then

$$W_2 = \{(v, Z_v \cap V(G_2)) \ : \ v \in V(G_2),\ (v, Z_v) \in W\} \cup \{(b, \{b\})\}$$

is a watching system for G_2.
If $(a, Z_a) \in L_W(b)$ and $L_W(b) \setminus \{(a, Z_a)\} \neq L_W(x)$ for every $x \in V(G_2) \setminus \{b\}$, then

$$W_2 = \{(v, Z_v \cap V(G_2)) \ : \ v \in V(G_2),\ (v, Z_v) \in W\}$$

is a watching system for G_2.
If $(a, Z_a) \notin L_W(b)$, then $W_2 = \{(v, Z_v \cap V(G_2)) \ : \ v \in V(G_2),\ (v, Z_v) \in W\}$ is a watching system for G_2. Thus $\omega(G_1) + \omega(G_2) \leq |W_1| + |W_2|$ or $\omega(G_1) + \omega(G_2) \leq |W_1| + |W_2| + 1$ or $\omega(G_1) + \omega(G_2) \leq |W_1| + |W_2| + 2$. However, $\omega(G_1) + \omega(G_2) - 2 \leq \omega(G)$. By Example 2.2, these two bounds are sharp. □

Following Ashrafi et al. [1], a link of graphs G_1 and G_2 by vertices $a \in V(G_1)$ and $b \in V(G_2)$ is defined as the graph $(G_1 \sim G_2)(a, b)$ obtained by joining a and b by an edge in the union of these graphs.

Corollary 2.5. *Let $G \simeq (K_{1,r} \sim K_{1,s})(a, b)$, where a and b be the universal vertices of $K_{1,r}$ and $K_{1,s}$, respectively. Then $\lceil log_2(\frac{r+2}{2})\rceil + \lceil log_2(\frac{s+2}{2})\rceil \leq \omega(G) \leq \lceil log_2(r + 2)\rceil + \lceil log_2(s + 2)\rceil$.*

Proof. By Theorems 2.3 and 2.4, the proof is straightforward. □

Theorem 2.6. *Let G be a connected graph of order n with $\Delta(G) = n - 2$, and let a and b be two distinct vertices in G such that $N_G(a) = N_G(b) = G \setminus \{a, b\}$. Then $\omega(G) = \lceil log_2(n + 1)\rceil$.*

Proof. By Theorem 2.1, (ii), $\lceil log_2(n+1)\rceil \leq \omega(G)$. If $n = 2^k - 1$, then $k \leq \omega(G)$. Let $x \in N_G(a)$ and $N_G(a) \setminus \{x\} = A \cup B$, where $A \cap B = \emptyset$ and $|A| = |B| = 2^{k-1} - 2$. Let induced subgraph on $A \cup \{a\}$ in G be A_a. By Theorem 2.3, $\omega(A_a) = \lceil log_2(2^{k-1} - 1 + 1)\rceil = k - 1$. Supoose $W_1 = \{\omega_i = (a, Z_i) \ : \ Z_i \subseteq N_{A_a}[a],\ 1 \leq i \leq k - 1\}$ be a watching system for A_a such that $a \in Z_i$ for every $1 \leq i \leq k - 1$. Since $|A| = |B|$, there exist a bijective function $f : A \longrightarrow B$.
Suppose that $W_2 = \{\omega_{2i} = (a, Z_i \cup f(Z_i) \cup \{x\}) \ : \ (a, Z_i) \in W_1,\ 1 \leq i \leq k-1\}$ and $W = W_2 \cup \{(b, Z_b)\}$, where $Z_b = B \cup \{x, b\}$. Then we have:

$$L_W(a) = \{\omega_{2i} \in W_2 \ : \ \omega_i \in L_{W_1}(a)\},$$

$$L_W(b) = \{\omega_b\},$$

$$L_W(x) = \{\omega_b\} \cup W_2,$$

$$L_W(y) = \{\omega_{2i} \in W_2 \ : \ \omega_i \in L_{W_1}(y)\}, \ L_W(f(y)) = L_W(y) \cup \{\omega_b\}, \quad \text{if } y \in A.$$

Hence $L_W(v) \neq \emptyset$ and $L_W(u) \neq L_W(v)$ for every u and v in $V(G)$. Thus W is a watching system for G. Hence, $\omega(G) \leq |W| = k$. Therefore, $\omega(G) = k = \lceil log_2(n+1) \rceil$.
Let $n \neq 2^k - 1$. Then $\lceil log_2(n+1) \rceil = \lceil log_2(n+2) \rceil$. By Theorem 2.1, (ii),

$$\lceil log_2(n+1) \rceil = \lceil log_2(n+2) \rceil \leq \omega(G).$$

Let n be even, $N_G(a) = A \cup B$, $A \cap B = \emptyset$ and $|A| = |B| = \frac{n-2}{2}$. Since $|A| = |B|$, there exist a bijective function $f : A \longrightarrow B$. By Theorem 2.3, $\omega(A_a) = \lceil log_2 \frac{n+2}{2} \rceil$. Let $W_1 = \{\omega_i = (a, Z_i) \ : \ Z_i \subseteq N_{A_a}[a], \ 1 \leq i \leq t\}$ be a watching system for A_a such that $a \in Z_i$ for every $1 \leq i \leq t$ and $t = \lceil log_2 \frac{n+2}{2} \rceil$. Suppose that $W_2 = \{\omega_{2i} = (a, Z_i \cup f(Z_i)) \ : \ (a, Z_i) \in W_1, \ 1 \leq i \leq t\}$ and $W = W_2 \cup \{(b, Z_b)\}$, where $Z_b = B \cup \{b\}$. Then we have:

$$L_W(a) = \{\omega_{2i} \in W_2 \ : \ \omega_i \in L_{W_1}(a)\},$$

$$L_W(b) = \{\omega_b\},$$

$$L_W(y) = \{\omega_{2i} \in W_2 \ : \ \omega_i \in L_{W_1}(y)\}, \ L_W(f(y)) = L_W(y) \cup \{\omega_b\}, \quad \text{if } y \in A.$$

Thus W is a watching system for G. Hence $\omega(G) \leq |W| = \lceil log_2 \frac{n+2}{2} \rceil + 1 = \lceil log_2(n+2) \rceil$. Therefore, $\omega(G) = \lceil log_2(n+2) \rceil = \lceil log_2(n+1) \rceil$.
Let n be odd, $x \in N_G(a)$ and $N_G(a) \setminus \{x\} = A \cup B$, $A \cap B = \emptyset$ and $|A| = |B| = \frac{n-3}{2}$. Since $|A| = |B|$, there exist a bijective function $f : A \longrightarrow B$. By Theorem 2.3, $\omega(A_a) = \lceil log_2 \frac{n+1}{2} \rceil$. Let $W_1 = \{\omega_i = (a, Z_i) \ : \ Z_i \subseteq N_{A_a}[a], \ 1 \leq i \leq t\}$ be a watching system for A_a such that $a \in Z_i$ for every $1 \leq i \leq t$ and $t = \lceil log_2 \frac{n+1}{2} \rceil$. Suppose that

$$W_2 = \{\omega_{2i} = (a, Z_i \cup f(Z_i) \cup \{x\}) \ : \ (a, Z_i) \in W_1, \ 1 \leq i \leq t\} \text{ and } W = W_2 \cup \{(b, Z_b)\},$$

where $Z_b = B \cup \{b, x\}$. Then we have:

$$L_W(a) = \{\omega_{2i} \in W_2 \ : \ \omega_i \in L_{W_1}(a)\},$$

$$L_W(b) = \{\omega_b\},$$

$$L_W(x) = \{\omega_b\} \cup W_2,$$

$$L_W(y) = \{\omega_{2i} \in W_2 \ : \ \omega_i \in L_{W_1}(y)\}, \ L_W(f(y)) = L_W(y) \cup \{\omega_b\}, \quad \text{if } y \in A.$$

Thus W is a watching system for G. Hence $\omega(G) \leq |W| = \lceil log_2(n+1) \rceil$. Therefore, $\omega(G) = \lceil log_2(n+1) \rceil$. □

Corollary 2.7. *Let G be an $(n-2)$-regular graph of order n. Then $\omega(G) = \lceil log_2(n+1) \rceil$.*

Proof. By Theorem 2.6, the proof is straightforward. □

DOI: https://dx.doi.org/10.30504/JIMS.2023.388523.1097

3. Watching systems and Mycielski's construction of graphs

In this Section, we consider the watching number of Mycielski's construction of some graphs.

Theorem 3.1. [2] *Let $n \geq 3$ be a positive integer. Then*

$$\omega(C_n) = \begin{cases} 3 & \textit{if } n = 4 \\ \lceil \frac{n}{2} \rceil & \textit{if } n \neq 4. \end{cases}$$

Theorem 3.2. *Let G be a graph of order n. Then*

$$\lceil log_2(n+1) \rceil + 1 \leq \omega(\mu(G)) \leq \omega(G) + \lceil log_2(n+2) \rceil.$$

Furthermore, these bounds are sharp.

Proof. It is clear that $|V(\mu(G))| = 2n+1$. By Theorem 2.1, (ii),

$$\omega(\mu(G)) \geq \lceil log_2(2n+1) + 1 \rceil = \lceil log_2(2n+2) \rceil = \lceil log_2(n+1) \rceil + 1.$$

Now, let $\omega(G) = k$ and $W_1 = \{\omega_1, \omega_2, \ldots, \omega_k\}$ be a watching system of G, where $\omega_i = (a_i, Z_i)$, $a_i \in V(G)$ and $Z_i \subseteq N_G[a_i]$. By definition of Mycilski's construction, induced subgraph on $V'(G) \cup \{w\}$ in $\mu(G)$ is isomorphic to $K_{1,n}$. By Theorem 2.3, $\omega(K_{1,n}) = \lceil log_2(n+2) \rceil$. For $1 \leq i \leq t = \lceil log_2(n+2) \rceil$, suppose that $W_2 = \{{\omega_1}', {\omega_2}', \ldots, {\omega_t}'\}$ be a watching system for induced subgraph on $V'(G) \cup \{w\}$ in $\mu(G)$, where $\omega_i' = (w, T_i')$ and $T_i' \subseteq N_{\mu(G)}[w]$. Let $W = W_1 \cup W_2$. Then for every $x \in V(G)$, we have $L_W(x) = L_{W_1}(x)$, $L_W(x') = L_{W_2}(x')$ and $L_W(w) = L_{W_2}(w)$, where x' is the copy of x in $\mu(G)$. So for every $y \in V(\mu(G))$ the sets $L_W(y)$ are non-empty and distinct. Hence W is a watching system of $\mu(G)$. Therefore, $\omega(\mu(G)) \leq |W| = |W_1| + |W_2| = \omega(G) + \lceil log_2(n+2) \rceil$. Therefore,

$$\lceil log_2(n+1) \rceil + 1 \leq \omega(\mu(G)) \leq \omega(G) + \lceil log_2(n+2) \rceil.$$

We know that $\mu(P_2) = C_5$. By Theorem 3.1, $\omega(\mu(P_2)) = 3$. On the other hand, we have $\lceil log_2(n+1) \rceil + 1 = 3$. This shows that the lower bound is sharp. If $G \cong \overline{K_n}$, then $\omega(\mu(G)) = n + \lceil log_2(n+2) \rceil$. This shows that the upper bound is sharp. □

Theorem 3.3. *Let G be a graph of order n with $\Delta(G) = n-1$. Then*

$$\omega(\mu(G)) = \lceil log_2(n+1) \rceil + 2.$$

Proof. By Theorem 3.2, $\lceil log_2(n+1) \rceil + 1 \leq \omega(\mu(G))$. By Theorem 2.3, $\omega(G) = \lceil log_2(n+1) \rceil$. Let a be a universal vertex of G and $W_1 = \{(a, Z_i) \ : \ 1 \leq i \leq t, \ Z_i \subseteq N_G[a]\}$ be a watching system of G, where $t = \lceil log_2(n+1) \rceil$.

Let $W_2 = \{\omega_{2i} = (a, Z_i \cup Z_i') \ : \ 1 \leq i \leq t, \ Z_i \subseteq N_G[a]\} \cup \{(w, N_{\mu(G)}[w]), (w, \{w\})\}$, where $Z_i' \subseteq V'(G)$ is a copy of Z_i in $\mu(G)$. Then we have:

$$L_{W_2}(a) = \{\omega_{2i} \in W_2 \ : \ \omega_i \in L_{W_1}(a)\},$$

$$L_{W_2}(x) = \{\omega_{2i} \in W_2 \ : \ \omega_i \in L_{W_1}(x)\}, \text{ for every } x \in V(G) \setminus \{a\},$$

DOI: https://dx.doi.org/10.30504/JIMS.2023.388523.1097

$$L_{W_2}(x^{'}) = L_{W_2}(x) \cup \{(w, N_{\mu(G)}[w])\}, \text{ for every } x \in V(G) \setminus \{a\}, \ x^{'} \text{is the copy of } x,$$

$$L_{W_2}(a^{'}) = \{(w, N_{\mu(G)}[w])\},$$

$$L_{W_2}(w) = \{(w, N_{\mu(G)}[w]), (w, \{w\})\}.$$

Thus W_2 is a watching system for G and so $\omega(\mu(G)) \leq |W_2| = \lceil log_2(n+1) \rceil + 2$.

Now, let $\omega(\mu(G)) = \lceil log_2(n+1) \rceil + 1$ and W be a watching system of $\mu(G)$ with minimum cardinality. Then by Theorem 2.3, there are $\lceil log_2(n+1) \rceil$ watchers located at vertex a. This watching system must cover $a^{'}$ and w. So there exist a watcher ω_1 is located at $a^{'}$ or w. If $\omega_1 = (a^{'}, Z_{a^{'}})$ and $\{a^{'}, w\} \subseteq Z_{a^{'}} \subseteq N_{\mu(G)}[a^{'}]$, then $L_W(a^{'}) = \{\omega_1\}$ and $L_W(w) = \{\omega_1\}$ which is a contradiction. If $\omega_1 = (w, Z_w)$ and $\{a^{'}, w\} \subseteq Z_w \subseteq N_{\mu(G)}[w]$, then $L_W(a^{'}) = \{\omega_1\}$ and $L_W(w) = \{\omega_1\}$, which is impossible. Hence, $\omega(\mu(G)) \neq \lceil log_2(n+1) \rceil + 1$. Therefore, $\omega(\mu(G)) = \lceil log_2(n+1) \rceil + 2$. □

Theorem 3.4. *Let $s \geq 2$ and $G \cong K_{2,s}$. Then $\omega(G) = \lceil log_2(s+3) \rceil$ and $\omega(\mu(G)) = \omega(G) + 2$.*

Proof. By Theorem 2.6, $\omega(G) = \lceil log_2(s+3) \rceil$. Suppose that the bipartition of G be $X = \{a, b\}$ and Y with $|Y| = s$.

Let $t = \lceil log_2(s+3) \rceil$ and $W_1 = \{\omega_1 = (a, Z_1), \ldots, \omega_{t-1} = (a, Z_{t-1}), \omega_t = (b, Z_b)\}$ be a watching system for G according to proof of Theorem 2.6. Also, let $\omega_{2i} = (a, Z_i \cup Z_i^{'})$ for $1 \leq i \leq t-1$, $\omega_{2t} = (b, Z_b \cup Z_b^{'})$, $\omega_{a^{'}} = (a^{'}, N_{\mu(G)}[a^{'}])$, $\omega_{b^{'}} = (b^{'}, N_{\mu(G)}[b^{'}])$ and $W_2 = \{\omega_{2i} \ : \ 1 \leq i \leq t-1\} \cup \{\omega_{a^{'}}, \omega_{b^{'}}\}$, where $Z_i^{'}$, $a^{'}$ and $b^{'}$ are the copy of Z_i, a and b, respectively, in $\mu(G)$ (See Figure 2). Then we have:

$$L_{W_2}(a) = \{\omega_{2j} \in W_2 \ : \ \omega_j \in L_{W_1}(a)\},$$

$$L_{W_2}(b) = \{\omega_{2j} \in W_2 \ : \ \omega_j \in L_{W_1}(b)\},$$

$$L_{W_2}(y) = \{\omega_{2j} \in W_2 \ : \ \omega_j \in L_{W_1}(y)\} \cup \{\omega_{a^{'}}, \omega_{b^{'}}\}, \text{ for every } y \in Y,$$

$$L_{W_2}(y^{'}) = \{\omega_{2j} \in W_2 \ : \ \omega_j \in L_{W_1}(y)\}, \text{ for every } y^{'} \in Y^{'}, \ y^{'} \text{is the copy of } \ y,$$

$$L_{W_2}(a^{'}) = \{\omega_{a^{'}}\},$$

$$L_{W_2}(b^{'}) = \{\omega_{b^{'}}\},$$

$$L_{W_2}(w) = \{\omega_{a^{'}}, \omega_{b^{'}}\}.$$

Thus W_2 is a watching system for $\mu(G)$ and so $\omega(\mu(G)) \leq |W_2| = \lceil log_2(s+3) \rceil + 2$.

Now, suppose that $\omega(\mu(G)) \neq \lceil log_2(s+3) \rceil + 2$ and W be a watching system for $\mu(G)$ with minimum cardinality. Then by Theorem 2.6, $\lceil log_2(s+3) \rceil$ watchers must are located at two vertices a and b. Also, another watcher must is located at $a^{'}$, $b^{'}$ or w. Anyway, we will have, $L_W(a^{'}) = L_W(w)$ or $L_W(b^{'}) = L_W(w)$. It is impossible. So $\lceil log_2(s+3) \rceil + 2 \leq \omega(\mu(G))$. Therefore, $\omega(\mu(G)) = \lceil log_2(s+3) \rceil + 2$. □

Theorem 3.5. *Let $n \geq 5$ be a positive integer. Then*

1) *If n is odd, $\omega(\mu(C_n)) \leq \lceil \frac{n}{2} \rceil + \left\lceil log_2(\frac{n+1}{2}) \right\rceil$.*
2) *If n is even, $\omega(\mu(C_n)) \leq \frac{n}{2} + \left\lceil log_2(\frac{n+4}{2}) \right\rceil$.*

DOI: https://dx.doi.org/10.30504/JIMS.2023.388523.1097

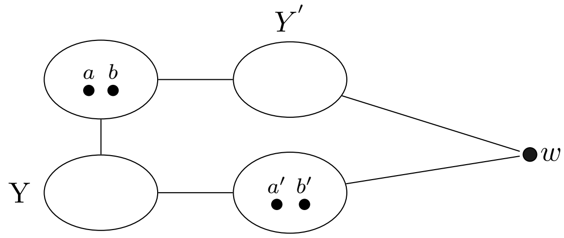


FIGURE 2. $\mu(K_{2,s})$

Bold line: Every vertex of the set is adjacent to every vertex of the other set.

Proof. 1) Let $n = 2k+1$, $V(\mu(C_n)) = \{v_i \ : \ 1 \leq i \leq n\} \cup \{v_i^{'} \ : \ 1 \leq i \leq n\} \cup \{w\}$, (See Figure 3), and H be induced subgraph on $\{v_{2i-1}^{'} \ : \ 2 \leq i \leq k\} \cup \{w\}$ in $\mu(C_n)$. By Theorem 2.3, $\omega(H) = \left\lceil log_2(\frac{n+1}{2})\right\rceil$. Let $W_1 = \{\omega_i = (w, Z_i) \ : \ Z_i \subseteq N_{\mu(C_n)}[w] \ , \ 1 \leq i \leq t\}$ be a watching system for H, where $t = \left\lceil log_2(\frac{n+1}{2})\right\rceil$.
Also, let $\omega_1^{'} = (w, Z_1 \cup \{v_1^{'}\} \cup \{v_{2j}^{'} \ : \ 1 \leq j \leq k\})$, $\omega_t^{'} = (w, Z_t \cup \{v_n^{'}\})$, $\omega_i^{'} = \omega_i$ for $2 \leq i \leq t-1$ and $W_1^{'} = \{\omega_1^{'}, \ldots, \omega_t^{'}\}$.
We claim that if $\omega_1^{''} = (v_1, \{v_1, v_2, v_2^{'}, v_n^{'}\})$ and $\omega_{\lceil \frac{n}{2}\rceil}^{''} = (v_n, \{v_1^{'}, v_{n-1}, v_n\})$, then

$$W = W_1^{'} \cup \{\omega_1^{''}, \ \omega_{\lceil \frac{n}{2}\rceil}^{''}\} \cup \{\omega_j^{''} = (v_{2j-1}, \{v_{2j-2}, v_{2j-1}, v_{2j}, v_{2j}^{'}\}) \ : \ 2 \leq j \leq k\},$$

is a watching system for $\mu(C_n)$. Because we have:

$$L_W(v_{2j-1}^{'}) = \{\omega_j^{'} \ : \ \omega_j \in L_{W_1}(v_{2j-1}^{'})\}, \ L_W(v_{2j-1}) = \{\omega_j^{''}\}, \ 2 \leq j \leq k,$$
$$L_W(v_{2j}^{'}) = \{\omega_1^{'}, \omega_j^{''}\}, \ 2 \leq j \leq k,$$
$$L_W(w) = \{\omega_j^{'} \ : \ \omega_j \in L_{W_1}(w)\},$$
$$L_W(v_1^{'}) = \{\omega_1^{'}, \omega_{\lceil \frac{n}{2}\rceil}^{''}\}, \ L_W(v_n^{'}) = \{\omega_t^{'}, \omega_1^{''}\},$$
$$L_W(v_2^{'}) = \{\omega_1^{'}, \omega_1^{''}\}, \ L_W(v_n) = \{\omega_{\lceil \frac{n}{2}\rceil}^{''}\},$$
$$L_W(v_1) = \{\omega_1^{''}\}, \ L_W(v_{2k}) = \{\omega_k^{''}, \omega_{\lceil \frac{n}{2}\rceil}^{''}\},$$
$$L_W(v_{2j}) = \{\omega_j^{''}, \omega_{j+1}^{''}\}, \ 1 \leq j \leq k-1.$$

Therefore, $\omega(\mu(C_n)) \leq \left\lceil \frac{n}{2}\right\rceil + \left\lceil log_2(\frac{n+1}{2})\right\rceil$.

2) Let $n = 2k$ and H be induced subgraph on $\{v_{2i-1}^{'} \ : \ 1 \leq i \leq k\} \cup \{w\}$ in $\mu(C_n)$. By Theorem 2.3, $\omega(H) = \left\lceil log_2(\frac{n+4}{2})\right\rceil$. Let $W_1 = \left\{\omega_i = (w, Z_i) \ : \ Z_i \subseteq N_{\mu(C_n)}[w] \ , \ 1 \leq i \leq t\right\}$ be a watching system for H, where $t = \left\lceil log_2(\frac{n+4}{2})\right\rceil$.
Also, let $\omega_1^{'} = (w, Z_1 \cup \{v_2^{'}, v_4^{'}, \ldots, v_{2k}^{'}\})$, $\omega_i^{'} = \omega_i$ for $2 \leq i \leq t$ and $W_1^{'} = \{\omega_1^{'}, \ldots, \omega_t^{'}\}$. We claim that $W = W_1^{'} \cup \{\omega_j^{''} = (v_{2j-1}, N_{C_n}[v_{2j-1}] \cup \{v_{2j}^{'}\}) \ : \ 1 \leq j \leq k\}$ is a watching system for $\mu(C_n)$. It is easy to see that:

$$L_W(v_{2j-1}^{'}) = \{\omega_j^{'} \ : \ \omega_j \in L_{W_1}(v_{2j-1}^{'})\}, \ L_W(v_{2j-1}) = \{\omega_i^{''}\}, \ 1 \leq j \leq k,$$
$$L_W(w) = \{\omega_j^{'} \ : \ \omega_j \in L_{W_1}(w)\},$$
$$L_W(v_{2j}^{'}) = \{\omega_1^{'}, \omega_j^{''}\}, \ 1 \leq j \leq k,$$

DOI: https://dx.doi.org/10.30504/JIMS.2023.388523.1097

$$L_W(v_{2j}) = \{\omega_j^{''}, \omega_{j+1}^{''}\},\ 1 \leq j \leq k-1,$$
$$L_W(v_{2k}) = \{\omega_1^{''}, \omega_k^{''}\}.$$

Therefore, $\omega(\mu(C_n)) \leq |W| = k + \left\lceil log_2(\frac{n+4}{2}) \right\rceil$.

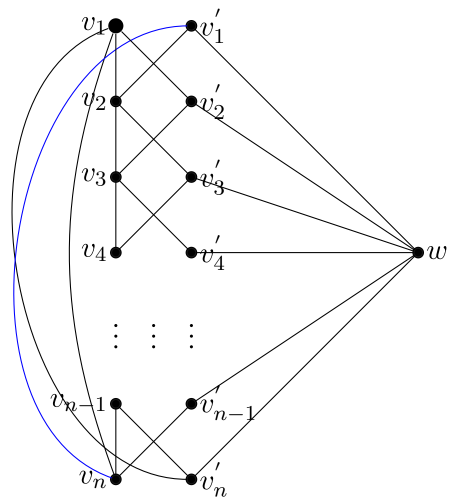


Figure 3. $\mu(C_n)$

□

Acknowledgements

The authors are very grateful to the referee for his/her useful comments.

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DOI: https://dx.doi.org/10.30504/JIMS.2023.388523.1097