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THE n-TH RESIDUAL RELATIVE OPERATOR ENTROPY $\mathfrak{R}^{[n]}_{x,y}(A|B)$ AND THE n-TH OPERATOR VALUED DIVERGENCE

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Abstract. We introduced the n-th residual relative operator entropy $\mathfrak{R}^{[n]}_{x,y}(A|B)$ and showed its monotone property, for example, $\mathfrak{R}^{[n]}_{x,x}(A|B) \leq \mathfrak{R}^{[n]}_{x,y}(A|B) \leq \mathfrak{R}^{[n]}_{y,y}(A|B)$ and $\mathfrak{R}^{[n]}_{x,x}(A|B) \leq \mathfrak{R}^{[n]}_{y,x}(A|B) \leq \mathfrak{R}^{[n]}_{y,y}(A|B)$ for $x \leq y$ if $A \leq B$ or n is odd. The n-th residual relative operator entropy $\mathfrak{R}^{[n]}_{x,y}(A|B)$ is not symmetric on x and y, that is, $\mathfrak{R}^{[n]}_{x,y}(A|B) \neq \mathfrak{R}^{[n]}_{y,x}(A|B)$ for $n \geq 2$ while $\mathfrak{R}^{[1]}_{x,y}(A|B) = \mathfrak{R}^{[1]}_{y,x}(A|B)$. In this paper we compare $\mathfrak{R}^{[n]}_{x,y}(A|B)$ with $\mathfrak{R}^{[n]}_{y,x}(A|B)$ and give the relations between $\mathfrak{R}^{[n]}_{x,y}(A|B)$ and the n-th operator divergence $\Delta^{[n]}_{i,x}(A|B)$. In this process, we find another operator divergence $\overline{\Delta}^{[n]}_{i,x}(A|B)$ which is similar to $\Delta^{[n]}_{i,x}(A|B)$ but not the same.

1. Introduction

Throughout this paper, the capital lettets A and B denote strictly positive operators on a Hilbert space and $A \natural_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}$ $(t \in \mathbb{R})$ is the path passing through A and B. For $t \in [0,1]$, it is an operator mean in the sense of Kubo and Ando [5] and is denoted by $A \sharp_t B$.

Fujii and Kamei [1] introduced the relative operator entropy as

$$S(A|B) \equiv \lim_{t\to 0} \frac{A \natural_t B - A}{t} = A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}.$$

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Furuta [2] generalized it to the generalized relative operator entropy as

$$S_x(A|B) \equiv \frac{d}{dt} A \natural_t B \bigg|_{t=x} = (A \natural_x B)A^{-1}S(A|B).$$

Yanagi, Kuriyama and Furuichi [8] introduced the Tsallis relative operator entropy as

$$T_x(A|B) \equiv \frac{A \natural_x B - A}{x} \quad (x \in \mathbb{R}\backslash\{0\}), \quad T_0(A|B) = S(A|B).$$

Among the above relative operator entropies we have inequalities as follows [3]:

$$S(A|B) \le T_x(A|B) \le S_x(A|B) \le -T_{1-x}(B|A) \le S_1(A|B) \quad (x \in (0,1)). \tag{1.1}$$

As differences between terms in (1.1), we defined operator valued divergences as follows:

$$\Delta_{1,x}(A|B) \equiv T_x(A|B) - S(A|B), \qquad \Delta_{2,x}(A|B) \equiv S_x(A|B) - T_x(A|B),$$

$$\Delta_{3,x}(A|B) \equiv -T_{1-x}(B|A) - S_x(A|B), \quad \Delta_{4,x}(A|B) \equiv S_1(A|B) + T_{1-x}(B|A).$$

As a generalization of $S(A|B)$ and $T_x(A|B)$, we introduced in [4] the n-th relative operator entropy $S^{[n]}(A|B)$, the n-th generalized relative operator entropy $S_x^{[n]}(A|B)$ and the n-th Tsallis relative operator entropy $T_x^{[n]}(A|B)$ as follows:

$$S^{[n]}(A|B) \equiv \frac{1}{n!}A(A^{-1}S(A|B))^n, \quad S_x^{[n]}(A|B) \equiv (A \natural_x B)A^{-1}S^{[n]}(A|B),$$

$$T_x^{[1]}(A|B) \equiv T_x(A|B) \quad \text{and}$$

$$T_x^{[n]}(A|B) \equiv \frac{T_x^{[n-1]}(A|B) - S^{[n-1]}(A|B)}{x} \ (x \neq 0), \quad T_0^{[n]}(A|B) \equiv S^{[n]}(A|B) \quad \text{for } n \ge 2.$$

Among them we have the following inequalities as (1.1) [4]: For $x \in [0,1]$

$$\begin{aligned} &S^{[n]}(A|B) \le T_x^{[n]}(A|B) \le S_x^{[n]}(A|B) \le T_{1-x}^{[n]}(B|A) \le S_1^{[n]}(A|B) \ \text{ for } n \text{ is odd or } A \le B, \\ &S^{[n]}(A|B) \ge T_x^{[n]}(A|B) \ge S_x^{[n]}(A|B) \ge T_{1-x}^{[n]}(B|A) \ge S_1^{[n]}(A|B) \ \text{ for } n \text{ is even and } A \ge B. \end{aligned} \tag{1.2}$$

Moreover we generalized $S^{[n]}(A|B)$, $S_x^{[n]}(A|B)$ and $T_x^{[n]}(A|B)$ to the n-th residual relative operator entropy $\mathfrak{R}_{x,y}^{[n]}(A|B)$ in [7] as follows:

$$\mathfrak{R}_{x,y}^{[n]}(A|B) \equiv A^{\frac{1}{2}}\Psi^{[n]}\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}, x, y\big)A^{\frac{1}{2}} \quad \text{for } x, y \in \mathbb{R},$$

where

$$\Psi^{[n]}(a, x, y) \equiv \frac{(\log a)^n}{(n-1)!}\int_0^1 (1-t)^{n-1}a^{(1-t)y+tx}dt \quad (a > 0).$$

We remark that $\mathfrak{R}_{0,0}^{[n]}(A|B) = S^{[n]}(A|B)$, $\mathfrak{R}_{x,x}^{[n]}(A|B) = S_x^{[n]}(A|B)$ and $\mathfrak{R}_{x,0}^{[n]}(A|B) = T_x^{[n]}(A|B)$ and have the following theorems.

Theorem 1.1. ([7, Theorem 2.2]) *Let A and B be strictly positive operators, $n \in \mathbb{N}$ and x, $y \in \mathbb{R}$. Then*

(a) *$\mathfrak{R}_{x,y}^{[n]}(A|B)$ is monotone increasing for each x and y if n is odd or $A \le B$,*

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(b) $\mathfrak{R}^{[n]}_{x,y}(A|B)$ *is monotone decreasing for each* x *and* y *if* n *is even and* $A \geq B$.

Theorem 1.2. ([7, Theorem 2.4]) *Let* A *and* B *be strictly positive operators,* $n \in \mathbb{N}$ *and* r, s, x, $y \in \mathbb{R}$. *Then*

$$\mathfrak{R}^{[n]}_{x,y}(A \natural_r B|A \natural_s B) = (s-r)^n \mathfrak{R}^{[n]}_{(1-x)r+xs,(1-y)r+ys}(A|B).$$

Theorem 1.3. ([7, Theorem 2.8]) *Let* A *and* B *be strictly positive operators,* $n \in \mathbb{N}$ *and* t, x, $y \in \mathbb{R}$. *Then*

$$(A \natural_t B)A^{-1}\mathfrak{R}^{[n]}_{x,y}(A|B) = \mathfrak{R}^{[n]}_{t+x,t+y}(A|B).$$

We know $\mathfrak{R}^{[1]}_{x,y}(A|B) = \mathfrak{R}^{[1]}_{y,x}(A|B)$ but $\mathfrak{R}^{[n]}_{x,y}(A|B)$ does not equal $\mathfrak{R}^{[n]}_{y,x}(A|B)$ for $n \geq 2$ in general, so we compare $\mathfrak{R}^{[n]}_{x,y}(A|B)$ with $\mathfrak{R}^{[n]}_{y,x}(A|B)$ and estimate the difference $\mathfrak{R}^{[n]}_{x,y}(A|B) - \mathfrak{R}^{[n]}_{y,x}(A|B)$ in some cases in Section 2. In Section 3 we show a relation between $\mathfrak{R}^{[n]}_{x,y}(A|B)$ and $\Delta^{[n]}_{i,x}(A|B)$ which is defined in [6]. We also introduce the new n-th operator valued divergences and compare them with $\Delta^{[n]}_{i,x}(A|B)$.

2. Relation between $\mathfrak{R}^{[n]}_{x,y}(A|B)$ and $\mathfrak{R}^{[n]}_{y,x}(A|B)$

In this section, we investigate relations between $\mathfrak{R}^{[n]}_{x,y}(A|B)$ and $\mathfrak{R}^{[n]}_{y,x}(A|B)$. We begin with the next lemma.

Lemma 2.1. *Let* c, $k > 0$, $n \in \mathbb{N}$ *and* g *be a continuous odd function. If* $g(t) \leq 0$ *for* $t \geq 0$ *then*

$$\int_{-k}^{k} (c-t)^n g(t)dt \geq 0.$$

If moreover $g(t) < 0$ *for some interval in* $[0, k]$, *then* $\int_{-k}^{k} (c-t)^n g(t)dt > 0$.

Proof. Since

$$(c-t)^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} c^{n-i} t^i = \sum_{i:\text{even}} \binom{n}{i} c^{n-i} t^i - \sum_{i:\text{odd}} \binom{n}{i} c^{n-i} t^i,$$

we have

$$\int_{-k}^{k} (c-t)^n g(t)dt = \sum_{i:\text{even}} \binom{n}{i} c^{n-i} \int_{-k}^{k} t^i g(t)dt - \sum_{i:\text{odd}} \binom{n}{i} c^{n-i} \int_{-k}^{k} t^i g(t)dt$$

$$= -2 \sum_{i:\text{odd}} \binom{n}{i} c^{n-i} \int_{0}^{k} t^i g(t)dt \geq 0.$$

The second assertion is clear. □

First we show the next result concerning the corresponding function $\Psi^{[n]}(a,x,y)$ to $\mathfrak{R}^{[n]}_{x,y}(A|B)$.

Proposition 2.2. *Let* $a, x, y \in \mathbb{R}$ *with* $x \leq y$ *and* $n \in \mathbb{N}$.

(a) *If* $a \geq 1$ *or* n *is odd, then* $\Psi^{[n]}(a,x,y) - \Psi^{[n]}(a,y,x) \geq 0$.

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(b) *If* $0 < a \leq 1$ *and* n *is even, then* $\Psi^{[n]}(a,x,y) - \Psi^{[n]}(a,y,x) \leq 0$.

Proof. Since $\Psi^{[n]}(a,x,y) = \dfrac{(\log a)^n}{(n-1)!}\displaystyle\int_0^1 (1-t)^{n-1} a^{(1-t)y+tx} dt$, we have

$$\Psi^{[n]}(a,x,y) - \Psi^{[n]}(a,y,x) = \frac{(\log a)^n}{(n-1)!}\int_0^1 (1-t)^{n-1}\left(a^{(1-t)y+tx} - a^{(1-t)x+ty}\right) dt$$

$$\left(\text{by putting } s = t - \frac{1}{2}\right)$$

$$= \frac{(\log a)^n}{(n-1)!}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2} - s\right)^{n-1}\left(a^{(\frac{1}{2}-s)y+(\frac{1}{2}+s)x} - a^{(\frac{1}{2}-s)x+(\frac{1}{2}+s)y}\right) ds$$

$$= \frac{(\log a)^n a^{\frac{1}{2}(x+y)}}{(n-1)!}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2} - s\right)^{n-1}\left(a^{(x-y)s} - a^{(y-x)s}\right) ds.$$

Since $g(s) = a^{(x-y)s} - a^{(y-x)s}$ is a odd function and furthermore, $g(s) \leq 0$ if $a \geq 1$ and $g(s) \geq 0$ if $0 < a \leq 1$, we have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2} - s\right)^{n-1}\left(a^{(x-y)s} - a^{(y-x)s}\right) ds \geq 0 \quad \text{if } a \geq 1$$

and

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2} - s\right)^{n-1}\left(a^{(x-y)s} - a^{(y-x)s}\right) ds \leq 0 \quad \text{if } 0 < a \leq 1$$

for $n \geq 2$ by Lemma 2.1, and therefore, we have

$$\Psi^{[n]}(a,x,y) - \Psi^{[n]}(a,y,x) \geq 0 \quad \text{if } \ a \geq 1 \text{ or } n \text{ is odd},$$

$$\Psi^{[n]}(a,x,y) - \Psi^{[n]}(a,y,x) \leq 0 \quad \text{if } 0 < a \leq 1 \text{ and } n \text{ is even}.$$

It is clear that $\Psi^{[1]}(a,x,y) - \Psi^{[1]}(a,y,x) = 0$. □

We give a condition for $\mathfrak{R}^{[n]}_{x,y}(A|B) = \mathfrak{R}^{[n]}_{y,x}(A|B)$.

Theorem 2.3. *Let* A *and* B *be strictly positive operators,* $n \in \mathbb{N}$ *with* $n \geq 2$ *and* x, $y \in \mathbb{R}$ *with* $x \neq y$. *Then the following hold:*

$$\mathfrak{R}^{[n]}_{x,y}(A|B) = \mathfrak{R}^{[n]}_{y,x}(A|B) \quad \textit{iff} \quad A = B.$$

Proof. It suffices to show that $\Psi^{[n]}(a,x,y) - \Psi^{[n]}(a,y,x) = 0$ if and only if $a = 1$. By the proof of Proposition 2.2, we have

$$\Psi^{[n]}(a,x,y) - \Psi^{[n]}(a,y,x) = \frac{(\log a)^n a^{\frac{1}{2}(x+y)}}{(n-1)!}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2} - s\right)^{n-1}\left(a^{(x-y)s} - a^{(y-x)s}\right) ds.$$

If $a = 1$ then the above difference equals 0. On the contrary, assume $a \neq 1$. Then $a^{(x-y)s} - a^{(y-x)s} > 0$ on $(-\frac{1}{2}, \frac{1}{2})$ or $a^{(x-y)s} - a^{(y-x)s} < 0$ on $(-\frac{1}{2}, \frac{1}{2})$. By Lemma 2.1, we obtain $\Psi^{[n]}(a,x,y) - \Psi^{[n]}(a,y,x) \neq 0$ since $n - 1 \geq 1$. □

Proposition 2.2 implies the next theorem immediately.

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Theorem 2.4. *Let A and B be strictly positive operators, $x \leq y$ and $n \in \mathbb{N}$. Then the following inequalities hold:*

(a) *If $A \leq B$ or n is odd then*

$$0 \leq \mathfrak{R}_{x,y}^{[n]}(A|B) - \mathfrak{R}_{y,x}^{[n]}(A|B).$$

(b) *If $A \geq B$ and n is even then*

$$0 \geq \mathfrak{R}_{x,y}^{[n]}(A|B) - \mathfrak{R}_{y,x}^{[n]}(A|B).$$

Remark 2.5. *It holds that $\mathfrak{R}_{x,y}^{[1]}(A|B) = \mathfrak{R}_{y,x}^{[1]}(A|B)$.*

The monotone property of $\mathfrak{R}_{x,y}^{[n]}(A|B)$ implies the next theorem which estimate the other side of the difference $\mathfrak{R}_{x,y}^{[n]}(A|B) - \mathfrak{R}_{y,x}^{[n]}(A|B)$.

Theorem 2.6. *Let A and B be strictly positive operators, α, $\beta \in \mathbb{R}$ with $\alpha < \beta$ and $n \in \mathbb{N}$. Then the following inequalities hold for x, $y \in \mathbb{R}$ with $\alpha \leq x, y \leq \beta$:*

(a) *If $A \leq B$ or n is odd then*

$$\mathfrak{R}_{x,y}^{[n]}(A|B) - \mathfrak{R}_{y,x}^{[n]}(A|B) \leq \mathfrak{R}_{\beta,\beta}^{[n]}(A|B) - \mathfrak{R}_{\alpha,\alpha}^{[n]}(A|B).$$

(b) *If $A \geq B$ and n is even then*

$$\mathfrak{R}_{x,y}^{[n]}(A|B) - \mathfrak{R}_{y,x}^{[n]}(A|B) \geq \mathfrak{R}_{\beta,\beta}^{[n]}(A|B) - \mathfrak{R}_{\alpha,\alpha}^{[n]}(A|B).$$

3. Applications of $\mathfrak{R}_{x,y}^{[n]}(A|B)$ to the n-th operator divergences

We defined in [6] the n-th operator valued divergences $\Delta_{i,x}^{[n]}(A|B)$ for $i = 1, 2, 3, 4$ as follows:

$$\Delta_{1,x}^{[n]}(A|B) \equiv (A \natural_0 B)A^{-1}\mathscr{D}_x^{[n]}(A|B), \quad \Delta_{2,x}^{[n]}(A|B) \equiv -(A \natural_x B)A^{-1}\mathscr{D}_{-x}^{[n]}(A|B),$$

$$\Delta_{3,x}^{[n]}(A|B) \equiv (A \natural_x B)A^{-1}\mathscr{D}_{1-x}^{[n]}(A|B), \quad \Delta_{4,x}^{[n]}(A|B) \equiv -(A \natural_1 B)A^{-1}\mathscr{D}_{x-1}^{[n]}(A|B),$$

where $\mathscr{D}_x^{[n]}(A|B) \equiv T_x^{[n]}(A|B) - S^{[n]}(A|B)$. Among them $\Delta_{1,x}^{[n]}(A|B)$ and $\Delta_{4,x}^{[n]}(A|B)$ are the differences between the n-th relative operator entropies in (1.2), that is,

$$\Delta_{1,x}^{[n]}(A|B) = T_x^{[n]}(A|B) - S^{[n]}(A|B) \quad \text{and} \quad \Delta_{4,x}^{[n]}(A|B) = S_1^{[n]}(A|B) - (-1)^n T_{1-x}^{[n]}(B|A),$$

however,

$$\Delta_{2,x}^{[n]}(A|B) \neq S_x^{[n]}(A|B) - T_x^{[n]}(A|B) \quad \text{and} \quad \Delta_{3,x}^{[n]}(A|B) \neq (-1)^n T_{1-x}^{[n]}(B|A) - S_x^{[n]}(A|B) \quad \text{for} \quad n \geq 2$$

while $\Delta_{2,x}^{[1]}(A|B) = S_x(A|B) - T_x(A|B)$ and $\Delta_{3,x}^{[1]}(A|B) = -T_{1-x}(B|A) - S_x(A|B)$.

First we show that all of $\Delta_{i,x}^{[n]}(A|B)$ are the differences between the n-th residual relative operator entropy $\mathfrak{R}_{x,y}^{[n]}(A|B)$ which is a generalization of $S_x^{[n]}(A|B)$ and $T_x^{[n]}(A|B)$.

Theorem 3.1. *Let A and B be strictly positive operators, $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then the following equalities hold:*

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(a) $\Delta_{1,x}^{[n]}(A|B) = \mathfrak{R}_{x,0}^{[n]}(A|B) - \mathfrak{R}_{0,0}^{[n]}(A|B) = T_x^{[n]}(A|B) - S^{[n]}(A|B)$.

(b) $\Delta_{2,x}^{[n]}(A|B) = \mathfrak{R}_{x,x}^{[n]}(A|B) - \mathfrak{R}_{0,x}^{[n]}(A|B)$.

(c) $\Delta_{3,x}^{[n]}(A|B) = \mathfrak{R}_{1,x}^{[n]}(A|B) - \mathfrak{R}_{x,x}^{[n]}(A|B)$.

(d) $\Delta_{4,x}^{[n]}(A|B) = \mathfrak{R}_{1,1}^{[n]}(A|B) - \mathfrak{R}_{x,1}^{[n]}(A|B) = S_1^{[n]}(A|B) - (-1)^n T_{1-x}^{[n]}(B|A)$.

Proof. (a) is trivial. We have (b) $\sim$ (d) by Theorem 1.3 as follows:

(b) $\Delta_{2,x}^{[n]}(A|B) = -(A \natural_x B)A^{-1}\mathscr{D}_{-x}^{[n]}(A|B) = -(A \natural_x B)A^{-1}\left(\mathfrak{R}_{-x,0}^{[n]}(A|B) - \mathfrak{R}_{0,0}^{[n]}(A|B)\right)$

$= -\mathfrak{R}_{0,x}^{[n]}(A|B) + \mathfrak{R}_{x,x}^{[n]}(A|B)$.

(c) $\Delta_{3,x}^{[n]}(A|B) = (A \natural_x B)A^{-1}\mathscr{D}_{1-x}^{[n]}(A|B) = (A \natural_x B)A^{-1}\left(\mathfrak{R}_{1-x,0}^{[n]}(A|B) - \mathfrak{R}_{0,0}^{[n]}(A|B)\right)$

$= \mathfrak{R}_{1,x}^{[n]}(A|B) - \mathfrak{R}_{x,x}^{[n]}(A|B)$.

(d) $\Delta_{4,x}^{[n]}(A|B) = -(A \natural_1 B)A^{-1}\mathscr{D}_{x-1}^{[n]}(A|B) = -(A \natural_1 B)A^{-1}\left(\mathfrak{R}_{x-1,0}^{[n]}(A|B) - \mathfrak{R}_{0,0}^{[n]}(A|B)\right)$

$= -\mathfrak{R}_{x,1}^{[n]}(A|B) + \mathfrak{R}_{1,1}^{[n]}(A|B)$. $\square$

We showed in [7] that for $\alpha \in [0,1]$,

$$\mathfrak{R}_{0,0}^{[n]}(A|B) \leq \mathfrak{R}_{\alpha,0}^{[n]}(A|B) \leq \mathfrak{R}_{\alpha,\alpha}^{[n]}(A|B) \leq \mathfrak{R}_{\alpha,1}^{[n]}(A|B) \leq \mathfrak{R}_{1,1}^{[n]}(A|B) \tag{a1}$$

$$\mathfrak{R}_{0,0}^{[n]}(A|B) \leq \mathfrak{R}_{0,\alpha}^{[n]}(A|B) \leq \mathfrak{R}_{\alpha,\alpha}^{[n]}(A|B) \leq \mathfrak{R}_{1,\alpha}^{[n]}(A|B) \leq \mathfrak{R}_{1,1}^{[n]}(A|B) \tag{a2}$$

if n is odd or $A \leq B$ and

$$\mathfrak{R}_{0,0}^{[n]}(A|B) \geq \mathfrak{R}_{\alpha,0}^{[n]}(A|B) \geq \mathfrak{R}_{\alpha,\alpha}^{[n]}(A|B) \geq \mathfrak{R}_{\alpha,1}^{[n]}(A|B) \geq \mathfrak{R}_{1,1}^{[n]}(A|B) \tag{b1}$$

$$\mathfrak{R}_{0,0}^{[n]}(A|B) \geq \mathfrak{R}_{0,\alpha}^{[n]}(A|B) \geq \mathfrak{R}_{\alpha,\alpha}^{[n]}(A|B) \geq \mathfrak{R}_{1,\alpha}^{[n]}(A|B) \geq \mathfrak{R}_{1,1}^{[n]}(A|B) \tag{b2}$$

if n is even and $A \geq B$.

There are eight differences of two terms next to each other in the above inequalities (a1) and (a2) (or (b1) and (b2)). The n-th operator valued divergences $\Delta_{i,x}^{[n]}(A|B)$ ($i = 1, 2, 3, 4$) are the four of them. We denote the rest of them by $\overline{\Delta}_{1,x}^{[n]}(A|B)$, $\overline{\Delta}_{2,x}^{[n]}(A|B)$, $\overline{\Delta}_{3,x}^{[n]}(A|B)$ and $\overline{\Delta}_{4,x}^{[n]}(A|B)$ as follows and compare them with $\Delta_{i,x}^{[n]}(A|B)$.

Definition 3.2. *Let A and B be strictly positive operators, $n \in \mathbb{N}$ and $x \in \mathbb{R}$.*

(a) $\overline{\Delta}_{1,x}^{[n]}(A|B) \equiv \mathfrak{R}_{0,x}^{[n]}(A|B) - \mathfrak{R}_{0,0}^{[n]}(A|B)$.

(b) $\overline{\Delta}_{2,x}^{[n]}(A|B) \equiv \mathfrak{R}_{x,x}^{[n]}(A|B) - \mathfrak{R}_{x,0}^{[n]}(A|B)$.

(c) $\overline{\Delta}_{3,x}^{[n]}(A|B) \equiv \mathfrak{R}_{x,1}^{[n]}(A|B) - \mathfrak{R}_{x,x}^{[n]}(A|B)$.

(d) $\overline{\Delta}_{4,x}^{[n]}(A|B) \equiv \mathfrak{R}_{1,1}^{[n]}(A|B) - \mathfrak{R}_{1,x}^{[n]}(A|B)$.

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We know immediately that

$$\Delta_{1,x}^{[n]}(A|B) - \overline{\Delta}_{1,x}^{[n]}(A|B) = \Delta_{2,x}^{[n]}(A|B) - \overline{\Delta}_{2,x}^{[n]}(A|B)$$

and

$$\Delta_{3,x}^{[n]}(A|B) - \overline{\Delta}_{3,x}^{[n]}(A|B) = \Delta_{4,x}^{[n]}(A|B) - \overline{\Delta}_{4,x}^{[n]}(A|B).$$

By the results in Section 2 we obtain the orders between $\Delta_{i,x}^{[n]}(A|B)$ and $\overline{\Delta}_{i,x}^{[n]}(A|B)$.

Proposition 3.3. *Let A and B be strictly positive operators and $n \in \mathbb{N}$. Then the following inequalities hold:*

(a) *If $A \leq B$ or n is odd then*

$$\Delta_{1,x}^{[n]}(A|B) \geq \overline{\Delta}_{1,x}^{[n]}(A|B) \text{ for } x \leq 0, \quad \Delta_{1,x}^{[n]}(A|B) \leq \overline{\Delta}_{1,x}^{[n]}(A|B) \text{ for } x \geq 0,$$
$$\Delta_{2,x}^{[n]}(A|B) \geq \overline{\Delta}_{2,x}^{[n]}(A|B) \text{ for } x \leq 0, \quad \Delta_{2,x}^{[n]}(A|B) \leq \overline{\Delta}_{2,x}^{[n]}(A|B) \text{ for } x \geq 0,$$
$$\Delta_{3,x}^{[n]}(A|B) \leq \overline{\Delta}_{3,x}^{[n]}(A|B) \text{ for } x \leq 1, \quad \Delta_{3,x}^{[n]}(A|B) \geq \overline{\Delta}_{3,x}^{[n]}(A|B) \text{ for } x \geq 1,$$
$$\Delta_{4,x}^{[n]}(A|B) \leq \overline{\Delta}_{4,x}^{[n]}(A|B) \text{ for } x \leq 1, \quad \Delta_{4,x}^{[n]}(A|B) \geq \overline{\Delta}_{4,x}^{[n]}(A|B) \text{ for } x \geq 1.$$

(b) *If $A \geq B$ and n is even then*

$$\Delta_{1,x}^{[n]}(A|B) \leq \overline{\Delta}_{1,x}^{[n]}(A|B) \text{ for } x \leq 0, \quad \Delta_{1,x}^{[n]}(A|B) \geq \overline{\Delta}_{1,x}^{[n]}(A|B) \text{ for } x \geq 0,$$
$$\Delta_{2,x}^{[n]}(A|B) \leq \overline{\Delta}_{2,x}^{[n]}(A|B) \text{ for } x \leq 0, \quad \Delta_{2,x}^{[n]}(A|B) \geq \overline{\Delta}_{2,x}^{[n]}(A|B) \text{ for } x \geq 0,$$
$$\Delta_{3,x}^{[n]}(A|B) \geq \overline{\Delta}_{3,x}^{[n]}(A|B) \text{ for } x \leq 1, \quad \Delta_{3,x}^{[n]}(A|B) \leq \overline{\Delta}_{3,x}^{[n]}(A|B) \text{ for } x \geq 1,$$
$$\Delta_{4,x}^{[n]}(A|B) \geq \overline{\Delta}_{4,x}^{[n]}(A|B) \text{ for } x \leq 1, \quad \Delta_{4,x}^{[n]}(A|B) \leq \overline{\Delta}_{4,x}^{[n]}(A|B) \text{ for } x \geq 1.$$

We have relations similar to Theorem 3.5 in [6] concerning $\overline{\Delta}_{i,x}^{[n]}(A|B)$ we defined above.

Theorem 3.4. *Let A and B be strictly positive operators, $n \in \mathbb{N}$ and r, s, $x \in \mathbb{R}$. Then the following hold:*

(a) $\overline{\Delta}_{1,x}^{[n]}(A \natural_r B|A \natural_s B) = (s-r)^n (A \natural_r B) A^{-1} \overline{\Delta}_{1,(s-r)x}^{[n]}(A|B)$.

(b) $\overline{\Delta}_{2,x}^{[n]}(A \natural_r B|A \natural_s B) = -(s-r)^n (A \natural_{(1-x)r+xs} B) A^{-1} \overline{\Delta}_{1,-(s-r)x}^{[n]}(A|B)$.

(c) $\overline{\Delta}_{3,x}^{[n]}(A \natural_r B|A \natural_s B) = (s-r)^n (A \natural_{(1-x)r+xs} B) A^{-1} \overline{\Delta}_{1,(s-r)(1-x)}^{[n]}(A|B)$.

(d) $\overline{\Delta}_{4,x}^{[n]}(A \natural_r B|A \natural_s B) = -(s-r)^n (A \natural_s B) A^{-1} \overline{\Delta}_{1,(s-r)(x-1)}^{[n]}(A|B)$.

Proof. By Theorem 1.2 and Theorem 1.3, we have

(a)
$$\begin{aligned}\overline{\Delta}_{1,x}^{[n]}(A \natural_r B|A \natural_s B) &= \mathfrak{R}_{0,x}^{[n]}(A \natural_r B|A \natural_s B) - \mathfrak{R}_{0,0}^{[n]}(A \natural_r B|A \natural_s B)\\ &= (s-r)^n \left(\mathfrak{R}_{r,(1-x)r+xs}^{[n]}(A|B) - \mathfrak{R}_{r,r}^{[n]}(A|B)\right)\\ &= (s-r)^n (A \natural_r B) A^{-1} \left(\mathfrak{R}_{0,(s-r)x}^{[n]}(A|B) - \mathfrak{R}_{0,0}^{[n]}(A|B)\right)\\ &= (s-r)^n (A \natural_r B) A^{-1} \overline{\Delta}_{1,(s-r)x}^{[n]}(A|B).\end{aligned}$$

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(b) $\overline{\Delta}^{[n]}_{2,x}(A \natural_r B|A \natural_s B) = \mathfrak{R}^{[n]}_{x,x}(A \natural_r B|A \natural_s B) - \mathfrak{R}^{[n]}_{x,0}(A \natural_r B|A \natural_s B)$

$= (s-r)^n \left(\mathfrak{R}^{[n]}_{(1-x)r+xs,(1-x)r+xs}(A|B) - \mathfrak{R}^{[n]}_{(1-x)r+xs,r}(A|B)\right)$

$= -(s-r)^n(A \natural_{(1-x)r+xs} B)A^{-1}\left(\mathfrak{R}^{[n]}_{0,-(s-r)x}(A|B) - \mathfrak{R}^{[n]}_{0,0}(A|B)\right)$

$= -(s-r)^n(A \natural_{(1-x)r+xs} B)A^{-1}\overline{\Delta}^{[n]}_{1,-(s-r)x}(A|B).$

(c) $\overline{\Delta}^{[n]}_{3,x}(A \natural_r B|A \natural_s B) = \mathfrak{R}^{[n]}_{x,1}(A \natural_r B|A \natural_s B) - \mathfrak{R}^{[n]}_{x,x}(A \natural_r B|A \natural_s B)$

$= (s-r)^n \left(\mathfrak{R}^{[n]}_{(1-x)r+xs,s}(A|B) - \mathfrak{R}^{[n]}_{(1-x)r+xs,(1-x)r+xs}(A|B)\right)$

$= (s-r)^n(A \natural_{(1-x)r+xs} B)A^{-1}\left(\mathfrak{R}^{[n]}_{0,(s-r)(1-x)}(A|B) - \mathfrak{R}^{[n]}_{0,0}(A|B)\right)$

$= (s-r)^n(A \natural_{(1-x)r+xs} B)A^{-1}\overline{\Delta}^{[n]}_{1,(s-r)(1-x)}(A|B).$

(d) $\overline{\Delta}^{[n]}_{4,x}(A \natural_r B|A \natural_s B) = \mathfrak{R}^{[n]}_{1,1}(A \natural_r B|A \natural_s B) - \mathfrak{R}^{[n]}_{1,x}(A \natural_r B|A \natural_s B)$

$= (s-r)^n \left(\mathfrak{R}^{[n]}_{s,s}(A|B) - \mathfrak{R}^{[n]}_{s,(1-x)r+xs}(A|B)\right)$

$= -(s-r)^n(A \natural_s B)A^{-1}\left(\mathfrak{R}^{[n]}_{0,(s-r)(x-1)}(A|B) - \mathfrak{R}^{[n]}_{0,0}(A|B)\right)$

$= -(s-r)^n(A \natural_s B)A^{-1}\overline{\Delta}^{[n]}_{1,(s-r)(x-1)}(A|B).$ □

Corollary 3.5. *Let A and B be strictly positive operators, $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then the following relations hold:*

(a) $\overline{\Delta}^{[n]}_{2,x}(A|B) = -(A \natural_x B)A^{-1}\overline{\Delta}^{[n]}_{1,-x}(A|B).$

(b) $\overline{\Delta}^{[n]}_{3,x}(A|B) = (A \natural_x B)A^{-1}\overline{\Delta}^{[n]}_{1,1-x}(A|B).$

(c) $\overline{\Delta}^{[n]}_{4,x}(A|B) = -BA^{-1}\overline{\Delta}^{[n]}_{1,x-1}(A|B).$

This corollary shows that $\overline{\Delta}^{[n]}_{i,x}(A|B)$ have similar formulations to the definition of $\Delta^{[n]}_{i,x}(A|B)$.

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