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THE FIRST EIGENVALUE OF (p, q)-ELLIPTIC QUASILINEAR SYSTEM ALONG THE RICCI FLOW

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Abstract. In this paper we investigate the monotonicity of the first eigenvalue of (p, q)-elliptic quasilinear systems along the Ricci flow in both normalized and unnormalized conditions. In particular, we study the eigenvalue problem for this system in the case of Bianchi classes for 3-homogeneous manifolds.

1. **Introduction**

Studying the eigenvalues of geometric operators is a hot topic in studying Riemannian manifolds. It is already known that the spectrum of Laplacian and p-Laplacian on a compact Riemannian manifold M encodes important geometric information. For estimates of the spectrum of other geometric quantities of M, see [7, 8].

Recently, there is an increasing interest in studying flows, including the heat flow and Ricci flow (see [11, 15]) as well as the mean curvature flow (for example see [9]).

Generally solving the geometric flow equations are too difficult due to their nonlinearity. The short time existence of solutions is obtained by the parabolic or hyperbolic nature of the equations.

The Ricci flow was first introduced in Physics by Friedan [16] in a context of string theory and later by Hamilton [17] in Mathematics.

Let M be a manifold with a Riemannian metric g_0, the family $g(t)$ of Riemannian metrics on M is

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called an unnormalized Ricci flow when it satisfies the equations

$$\frac{\partial}{\partial t}g(t) = -2Ric\,(g(t)) \tag{1.1}$$

with the initial condition $g(0) = g_0$ where Ric is the Ricci tensor of $g(t)$.
Also one may consider the normalized Ricci flow as

$$\frac{\partial}{\partial t}g(t) = -2Ric\,(g(t)) + \frac{2r}{n}g, \quad g(0) = g_0, \tag{1.2}$$

where $r = \frac{\int_M Rd\mu}{\int_M d\mu}$ is the average of the scalar curvature.
The existence and uniqueness of the solution of this flow is proved by Hamilton in [17] and De Turk in [14]. Now consider $g(t)$ as a solution of the Ricci flow (1.1), and aply the normalization to get

$$\bar{g}(\bar{t}) = \psi(t)g(t), \quad \bar{t} = \int_0^t \psi(\nu)d\nu,$$

where $\frac{1}{\psi}\frac{\partial\psi}{\partial t} = \frac{2r}{n}$ and r is as in (1.2). In this case $\bar{g}(\bar{t})$ is the solution of the normalized Ricci flow (1.2).
A functional $\mathcal{F}\,(g)$ of Riemannian metrics g is said to be monotonic under the Ricci flow if $\mathcal{F}\,(g\,(t))$ is nondecreasing in t whenever $g\,(.)$ is a Ricci flow solution.
Monotonic quantities are an important tool for understanding Ricci flow. In particular, formally thinking of Ricci flow as a flow on the space of metrics, one way to get monotonic quantity is to think of Ricci flow as the gradient flow of a functional $\mathcal{F}$.
The functional $\mathcal{F}$ was introduced by Perelman in [23] for smooth function f and metric g as

$$\mathcal{F}\,(g, f) = \int_M \left(R + |\nabla f|^2\right) e^{-f} d\mu,$$

whose gradient flow differs from the Ricci flow by the action of diffeomorphisms. If $g\,(.)$ satisfies the Ricci flow equation and e^{-f} satisfies the conjugate or backward heat equation in terms of $g\,(.)$, then $\mathcal{F}\,(g\,(t)\,, f\,(t))$ is nondecreasing in t. Furthermore, it is constant in t if and only if $g\,(.)$ is a gradient steady soliton with associated function f. Minimizing $\mathcal{F}\,(g, f)$ over all functions f with $\int_M e^{-f} d\mu = 1$ gives the monotonic quantity $\lambda_1\,(g)$, which turns out to be the lowest eigenvalue of $-4\Delta + R$. Later Cao in [4] improved Perelman's results for the first eigenvalue of the operator $-4\Delta + cR$ where $c \geq \frac{1}{4}$.
Although the Ricci flow was introduced by Hamilton for the first time it was Perelman's outstanding view of the first eigenvalue of the operator $-4\Delta + R$ which resulted in the current interests in Ricci flow.

2. Priliminaries and (p, q)-elliptic system

Consider M as a compact Riemannian manifold and let $u : M \longrightarrow \mathbb{R}$ be a smooth function on M. The p-Laplacian of u for $1 < p < \infty$ is defined as

$$\begin{aligned}\Delta_p u &= div(|\nabla u|^{p-2}\nabla u)\\ &= |\nabla u|^{p-2}\Delta u + (p-2)|\nabla u|^{p-4}(Hess\,u)(\nabla u, \nabla u),\end{aligned}$$

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where

$$\begin{aligned}(Hess\,u)\,(X,Y) &= \nabla\,(\nabla u)\,(X,Y)\\ &= X.\,(Y.u) - (\nabla_X Y)\,.u \quad X,Y \in \chi\,(M)\,.\end{aligned}$$

Now let (M^n, g) be a closed Riemannian manifold. In this paper we are going to study the (p,q)-elliptic quasilinear system

$$\text{(2.1)} \qquad \begin{cases} -\Delta_p u = \lambda u|u|^{p-2} + \lambda |u|^{\alpha}|v|^{\beta} v & \text{in M },\\ -\Delta_q v = \lambda v|v|^{q-2} + \lambda |u|^{\alpha}|v|^{\beta} u & \text{in M },\\ u = v = 0 & \text{on } \partial\text{M}, \end{cases}$$

where $p > 1$, $q > 1$ and α, β are real numbers such that

$$\alpha > 0,\ \beta > 0, \qquad \frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1.$$

In this situation λ is called an eigenvalue of system (2.1) and (u,v) are eigenfunctions corresponding to λ.

The first nonzero eigenvalue of the (p,q)-elliptic quasilinear system (2.1) is defined as

$$\lambda(t) = \inf_{u,v\neq 0}\{\frac{\alpha+1}{p}\int_M |\nabla u|^p d\mu + \frac{\beta+1}{q}\int_M |\nabla v|^q d\mu; \quad \Lambda = 1\},$$

where

$$\Lambda = \int_M |u|^{\alpha+1}|v|^{\beta+1} d\mu + \frac{\alpha+1}{p}\int_M |u|^p d\mu + \frac{\beta+1}{q}\int_M |v|^q d\mu.$$

the special case of the system (2.1) as

$$\begin{cases} \Delta_p u = -\lambda |u|^{\alpha}|v|^{\beta} v & \text{in M },\\ \Delta_q v = -\lambda |u|^{\alpha}|v|^{\beta} u & \text{in M },\\ u = v = 0 & \text{on } \partial\text{M}, \end{cases}$$

was studied before, for example in [2].

In monotonicity for the first eigenvalue of the system (2.1), it is important to make sure about the differentiability of the eigenvalue function. For this purpose we apply techniques of Cao and Wang in [4] and [25]. Let $t_0 \in [0,T)$, $(u_0, v_0) = (u(t_0), v(t_0))$ be the eigenfunctions for the eigenvalue $\lambda(t_0)$ of (p,q)-elliptic quasilinear system (2.1). We consider the smooth functions as

$$h(t) = u_0\left(\frac{det[g_{ij}(t)]}{det[g_{ij}(t_0)]}\right),\ l(t) = v_0\left(\frac{det[g_{ij}(t)]}{det[g_{ij}(t_0)]}\right),$$

along the Ricci flow. Now let

$$u(t) = \frac{h(t)}{G^{\frac{1}{p}}}, \quad v(t) = \frac{l(t)}{G^{\frac{1}{q}}},$$

where $u(t)$ and $v(t)$ are smooth functions under the Ricci flow,

$$G = \int_M |h(t)|^{\alpha}|l(t)|^{\beta} h(t) l(t) d\mu + \frac{\alpha+1}{p}\int_M |h(t)|^p d\mu + \frac{\beta+1}{q}\int_M |l(t)|^q d\mu,$$

and satisfy the equality

$$\int_M |u|^{\alpha+1}|v|^{\beta+1}d\mu + \frac{\alpha+1}{p}\int_M |u|^p d\mu + \frac{\beta+1}{q}\int_M |v|^q d\mu = 1. \tag{2.2}$$

At time t_0, $(u(t_0), v(t_0))$ is the eigenfunctions for $\lambda(t_0)$ of (p,q)-elliptic quasilinear system (2.1), and

$$\lambda(u, v, t_0) = \lambda(t_0),$$

for which we are going to give an evolution formula.

If $(M^n, g(t))$ is a solution of the Ricci flow (1.1) on the smooth manifold (M^n, g_0) in the interval $[0,T)$ then we may write the eigenvalue function $\lambda(u,v,t)$ as below

$$\lambda(u,v,t) = \frac{\alpha+1}{p}\int_M |\nabla u|^p d\mu + \frac{\beta+1}{q}\int_M |\nabla v|^q d\mu, \tag{2.3}$$

where u and v satisfy in (2.2).

N. Zographopoulos in [27] has discussed existence and uniqueness of solution of the (p,q)-elliptic quasilinear system (2.1). This type of systems appear in different cases in Physics. For example in the study of transport of electron temperature in a confined plasma and in the study of electromagnetic phenomena in nonhomogeneous super conductors, see [3,13]. For more details in electrochemistry and nuclear reaction, see [10] or [12].

Let us recall some results on the monotonic quantities of the first eigenvalue of systems including geometric operators.

Theorem 2.1 (Wu et al, [26])**.** *Let $g(t)$, $t \in [0,T)$, be a solution of the Ricci flow on a closed manifold M^n and $\lambda_{1,p}(t)$ be the first eigenvalue of the p-Laplace operator ($p > 1$) of $g(t)$. If there exists a non-negative constant ϵ such that*

$$R_{ij} - \frac{R}{p}g_{ij} \geq -\epsilon g_{ij},$$

and

$$R \geq p\epsilon,$$

then $\lambda_{1,p}(t)$ is strictly increasing and differentiable almost everywhere along the Ricci flow on $[0,T)$.

Theorem 2.2 (Azami [2])**.** *Let $(M^n, g(t))$ be a solution of the unnormalized mean curvature flow on the smooth closed manifold (M^n, g_0) and $\lambda(t)$ be the evolution of the first eigenvalue under the mean curvature flow. Let $n \leq k = \min\{p,q\}$. At the initial time $t = 0$, if $H > 0$, and there exists a non-negative constant ϵ such that*

$$h_{ij} \geq \epsilon H g_{ij} \quad \textit{and} \quad \frac{1}{k} \leq \epsilon \leq \frac{1}{n},$$

then $\lambda(t)$ is nondecreasing under the unnormalized mean curvature flow.

For more results along this line, we refer the reader to [1,20,21,24]. In this paper, we are going to improve the above results for the first eigenvalue of the system (2.1) under the Ricci flow. As a quick review, we list our main results as follows:

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Theorem 2.3. *Let $(M, g(t))$, $t \in [0,T)$ be a solution of the unnormalized Ricci flow (1.1), on a smooth closed Riemannian manifold (M^n, g_0) with nonnegative scalar curvature such that $R_{ij} - \gamma R g_{ij} \geq 0$ where $\gamma \geq \frac{1}{q}$ and also $p \geq q$. If $\lambda(t)$ denotes the evolution of the first eigenvalue of (p,q)-elliptic quasilinear system (2.1), then*

$$\lambda(t)\left(c - \frac{2}{n}t\right)^{\frac{n}{2}\left(\gamma + \frac{q-1}{q}\right)},$$

is increasing along the unnormalized Ricci flow, where $c = \frac{1}{R_{min}(0)}$.

Remark 2.4. *Consider $\left(M^3, g(t)\right)$, $t \in [0,T)$ as a solution of the Ricci flow (1.1) on a closed Riemannian manifold $\left(M^3, g_0\right)$. If there exist constant ϵ such that $\epsilon R g_{ij} \leq R_{ij}$ where $0 < \epsilon \leq \frac{1}{3}$ at time $t = 0$ on M, then this inequality is preserved by the flow (1.1) on $[0,T)$. Therefore, if Ricci curvature of $\left(M^3, g_0\right)$ is positive then the result of Theorem 2.3 holds for $\left(M^3, g(t)\right)$ with theorem's assumption.*

Theorem 2.5. *Consider $(M, g(t))$, $t \in [0,T)$ as a solution of the normalized Ricci flow (1.2), on a smooth closed Riemannian manifold (M^n, g_0) with nonnegative scalar curvature such that $R_{ij} - \gamma R g_{ij} \geq 0$ where $\gamma \geq \frac{1}{q}$ and also $p \geq q$. If $\lambda(t)$ denotes the evolution of the first eigenvalue of (p,q)-elliptic quasilinear system (2.1), then*

$$\lambda(t) e^{\frac{rq}{n}t}\left(2R_{min}(0)e^{-\frac{r}{n}t} + (r - 2R_{min}(0))\right)^{\frac{n}{2}\left(\gamma + \frac{q-1}{q}\right)}$$

is increasing along the normalized Ricci flow.

Theorem 2.6. *Let $(M, g(t))$, $t \in [0,T)$ be a solution of the Ricci flow (1.1) on a compact Riemannian manifold (M^n, g_0). If $\lambda(t)$ denotes the first positive eigenvalue of the (p,q)-elliptic quasilinear system (2.1) and*

$$I = \int_M u^2 |u|^{2p-4} d\mu + \int_M |u|^{2\alpha}|v|^{2\beta} v^2 d\mu + \int_M 2|u|^{p-2}|u|^{\alpha}|v|^{\beta} uv d\mu < \infty$$

then $\lambda(t) \longrightarrow \infty$ in finite time where $R_{ij} - \gamma R g_{ij} \geq 0$ in $M \times [0,T)$ and γ is a positive real constant.

Theorem 2.7. *Consider $(M, g(t))$, $t \in [0,T)$ as a solution of the normalized Ricci flow (1.2), on a smooth closed oriented Riemannian manifold $\left(M^2, g_0\right)$, and also $p \geq q \geq 2$. If $\lambda(t)$ denotes the evolution of the first eigenvalue of (p,q)-elliptic quasilinear system (2.1) then*

- *If $R < 0$ then*

$$\lambda(t) e^{\{-\left(\frac{p}{2} + \frac{q}{n}\right)rt + \frac{pc}{2r}e^{rt}\}},$$

is nondecreasing along the normalized Ricci flow (1.2), and

$$\lambda(t) e^{\{-\left(\frac{q}{2} + \frac{p}{n}\right)rt - \frac{pc}{2r}e^{rt}\}},$$

is nonincreasing along the normalized Ricci flow (1.2), where c is real constant.

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- *If $R > 0$ then*

$$\lambda(t)e^{\{\frac{qc}{2r}e^{rt}-\frac{qr}{n}t\}},$$

is nondecreasing along the normalized Ricci flow (1.2), and

$$\lambda(t)e^{\{-(\frac{p}{2}+\frac{q}{n})rt-\frac{pc}{2r}e^{rt}\}},$$

is nonincreasing along the normalized Ricci flow (1.2), where c is real constant.

3. Monotonicity and evolution formulas

In this section we prove the main results of this paper. First of all we are going to give evolution formulas in both normal and unnormal cases for the first eigenvalue of (p,q)-elliptic quasilinear system (2.1).

Proposition 3.1. *Let $(M, g(t))$ be a solution of the unnormalized Ricci flow (1.1) on a smooth closed Riemannian manifold (M^n, g_0). If $\lambda(t)$ denotes the first eigenvalue of the (p,q)-elliptic quasilinear system (2.1) then*

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} &= \lambda(t_0)\int_M R|u|^\alpha|v|^\beta uv d\mu + \frac{\alpha+1}{p}\lambda(t_0)\int_M R|u|^p d\mu \\ &+ \frac{\beta+1}{q}\lambda(t_0)\int_M R|v|^q d\mu + (\alpha+1)\int_M Ric\left(\nabla u,\nabla u\right)|\nabla u|^{p-2}d\mu \\ &+ (\beta+1)\int_M Ric\left(\nabla v,\nabla v\right)|\nabla v|^{q-2}d\mu - \frac{\alpha+1}{p}\int_M R|\nabla u|^p d\mu \\ &- \frac{\beta+1}{q}\int_M R|\nabla v|^q d\mu.\end{aligned}$$

Proof. As mentioned before, $\lambda(u,v,t)$ is differentiable, so differentiating the formula (2.3) with respect to the time variable t, we get

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} &= \frac{\alpha+1}{2}\Big[\int_M \Big\{-g^{ij}g^{jk}\frac{\partial}{\partial t}(g_{lk})\nabla_i u\nabla_j u + 2 < \nabla u_t, \nabla u > \Big\}|\nabla u|^{p-2}d\mu\Big] \\ &+ \frac{\beta+1}{2}\Big[\int_M \Big\{-g^{ij}g^{jk}\frac{\partial}{\partial t}(g_{lk})\nabla_i v\nabla_j v + 2 < \nabla v_t, \nabla v > \Big\}|\nabla v|^{q-2}d\mu\Big] \\ &+ \frac{\alpha+1}{p}\int_M |\nabla u|^p\Big[\frac{1}{2}tr_g(\frac{\partial g}{\partial t})d\mu\Big] \\ &+ \frac{\alpha+1}{q}\int_M |\nabla v|^q\Big[\frac{1}{2}tr_g(\frac{\partial g}{\partial t})d\mu\Big],\end{aligned} \tag{3.1}$$

where $u_t = \frac{\partial u}{\partial t}$. Also differentiating both sides of normalization condition

$$\int_M |u|^\alpha|v|^\beta uv d\mu + \frac{\alpha+1}{p}\int_M |u|^p d\mu + \frac{\beta+1}{q}\int_M |v|^\beta d\mu = 1,$$

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we have

$$\begin{aligned}0 &= \frac{d}{dt}\Big[\int_M |u|^\alpha |v|^\beta uv d\mu + \frac{\alpha+1}{p}\int_M |u|^p d\mu + \frac{\beta+1}{q}\int_M |v|^q d\mu\Big]\\ &= (\alpha+1)\int_M |u|^\alpha |v|^\beta u_t.v d\mu + \int_M |u|^\alpha |v|^\beta uv\left(\frac{1}{2}tr_g\left(\frac{\partial g}{\partial t}\right)\right)d\mu\\ &+ (\beta+1)\int_M |u|^{\alpha+1}|v|^\beta u.v_t d\mu + (\alpha+1)\int_M |u|^{p-2}u.u_t d\mu\\ &+ (\beta+1)\int_M |v|^{q-2}v.v_t d\mu + \frac{\alpha+1}{p}\int_M |u|^p\left(\frac{1}{2}tr_g\left(\frac{\partial g}{\partial t}\right)\right)d\mu\\ &+ \frac{\beta+1}{q}\int_M |v|^q\left(\frac{1}{2}tr_g\left(\frac{\partial g}{\partial t}\right)\right)d\mu,\end{aligned}$$

thus

$$\begin{aligned}&(\alpha+1)\int_M |u|^\alpha|v|^\beta u_t v d\mu + (\beta+1)\int_M |u|^{\alpha+1}|v|^\beta u v_t d\mu\\ &+ (\alpha+1)\int_M |u|^{p-2}u.u_t d\mu + (\beta+1)\int_M |v|^{q-2}v.v_t d\mu\\ &= \int_M R|u|^\alpha|v|^\beta uv d\mu + \frac{\alpha+1}{p}\int_M R|u|^p d\mu + \frac{\beta+1}{q}\int_M R|v|^q d\mu.\end{aligned}$$

Multiplying both sides of the (p,q)-elliptic quasilinear system (2.1) by u_t and v_t respectively we get

$$\begin{aligned}&(\alpha+1)\int_M <\nabla u_t, \nabla u> |\nabla u|^{p-2}d\mu + (\beta+1)\int_M <\nabla v_t,\nabla v> |\nabla v|^{q-2}d\mu\\ &= \lambda(t_0)\int_M R|u|^\alpha|v|^\beta uv d\mu + \frac{\alpha+1}{p}\lambda(t_0)\int_M R|u|^p d\mu\\ &+ \frac{\beta+1}{q}\lambda(t_0)\int_M R|v|^q d\mu.\end{aligned}$$

Now aplying the Ricci flow equation into (3.1), we get the resul. □

Similarly in the case of normalized Ricci flow (1.2) we have

Proposition 3.2. *Let $(M, g(t))$ be a solution of the normalized Ricci flow (1.2) on a smooth closed Riemannian manifold (M^n, g_0). If $\lambda(t)$ denotes the first eigenvalue of the (p,q)-elliptic quasilinear system (2.1) then*

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} &= \lambda(t_0)\int_M R|u|^\alpha|v|^\beta uv d\mu + \frac{\alpha+1}{p}\lambda(t_0)\int_M R|u|^p d\mu\\ &+ \frac{\beta+1}{q}\lambda(t_0)\int_M R|v|^q d\mu + (\alpha+1)\int_M Ric\left(\nabla u,\nabla u\right)|\nabla u|^{p-2}d\mu\\ &+ (\beta+1)\int_M Ric\left(\nabla v,\nabla v\right)|\nabla v|^{q-2}d\mu - \frac{\alpha+1}{p}\int_M R|\nabla u|^p d\mu\\ &- \frac{\beta+1}{q}\int_M R|\nabla v|^q d\mu + \frac{\alpha+1}{n}r\int_M |\nabla u|^p d\mu\\ &+ \frac{\beta+1}{n}r\int_M |\nabla v|^q d\mu.\end{aligned}$$

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Proof. Similar to Proposition 3.1, by replacing the normalized Ricci flow equation (1.2) into the formula (3.1) we get what we want. □

Now we are ready to prove our main theorems.

Proof of theorem 2.3. By Proposition 3.1 since $R_{ij} - \gamma R g_{ij} \geq 0$ we have

$$\begin{aligned} \frac{d}{dt}\lambda(u,v,t)|_{t=t_0} \geq \lambda(t_0)\int_M R|u|^\alpha|v|^\beta uv d\mu + \frac{\alpha+1}{p}\lambda(t_0)\int_M R|u|^p d\mu \\ + \frac{\beta+1}{q}\lambda(t_0)\int_M R|v|^q d\mu + (\alpha+1)\left(\gamma - \frac{1}{p}\right)\int_M |\nabla u|^p R d\mu \\ + (\beta+1)\left(\gamma - \frac{1}{q}\right)\int_M |\nabla v|^q R d\mu, \end{aligned}$$

and since $p \geq q$ we get

$$\begin{aligned} \frac{d}{dt}\lambda(u,v,t)|_{t=t_0} \geq \lambda(t_0)\int_M R|u|^\alpha|v|^\beta uv d\mu \\ + \frac{\alpha+1}{p}\lambda(t_0)\int_M R|u|^p d\mu + \frac{\beta+1}{q}\lambda(t_0)\int_M R|v|^q d\mu \\ + \left(\gamma - \frac{1}{q}\right)\Big[\frac{\alpha+1}{p}\int_M |\nabla u|^p R d\mu + \frac{\beta+1}{q}\int_M |\nabla v|^q R d\mu\Big]. \end{aligned}$$

Evolution of the scalar curvature R under the Ricci flow (1.1) is

$$\frac{\partial R}{\partial t} = \Delta R + 2|Ric|^2,$$

where by $|Ric|^2 \geq \frac{1}{n}R^2$ we get

$$\frac{\partial R}{\partial t} \geq \Delta R + \frac{2}{n}R^2.$$

Since the solution to the ODE $\frac{dy}{dt} = \frac{2}{n}y^2$ is

$$y(t) = \frac{1}{c - \frac{2}{n}t},$$

where $c = \frac{1}{R_{min}(0)}$ is a real constant and $y(0) \leq R(x,0)$, then the maximum principle yields

$$y(t) \leq R(x,t).$$

This implies that

$$\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} \geq \left(\gamma + \frac{q-1}{q}\right) y(t_0)\lambda(t_0),$$

and so, in any sufficiently small neighborhood of t_0 as I we get $\frac{d}{dt}\lambda(u,v,t) > \left(\gamma + \frac{q-1}{q}\right)\lambda(u,v,t)y(t)$, for any t_1 sufficiently close to t_0 where $[t_1, t_0] \subseteq I$,

$$\lambda(u,v,t_0) = \lambda(t_0), \quad \lambda(u,v,t_1) \geq \lambda(t_1), \tag{3.2}$$

and

$$ln\frac{\lambda(t_0)}{\lambda(t_1)} \geq \frac{n}{2}\left(\gamma + \frac{q-1}{q}\right) ln\frac{c - \frac{2}{n}t_1}{c - \frac{2}{n}t_0},$$

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since the function ln is increasing and t_0 is arbitrary, this implies what we are looking for. □

Proof of Theorem 2.5. Similar to the proof of Theorem 2.3, consider the evolution formula of R as

$$\frac{\partial}{\partial t}R = \Delta R + 2|Ric|^2 - \frac{r}{n}R,$$

where by $|Ric|^2 \geq \frac{1}{n}R^2$ we have

$$\frac{\partial}{\partial t}R \geq \Delta R + \frac{2}{n}R^2 - \frac{r}{n}R,$$

since the solution to the ODE $\frac{dy}{dt} = \frac{2}{n}y^2 - \frac{r}{n}y$ is

$$y(t) = \left(\frac{2}{r} + \left(\frac{1}{R_{min}(0)} - \frac{2}{r}\right)e^{\frac{r}{n}t}\right)^{-1}.$$

Now since $R_{ij} - \gamma R g_{ij} \geq 0$, $p \geq q$ by the maximum principle we get

$$\frac{d}{dt}\lambda(u, v, t)|t = t_0 \geq \left(\gamma + \frac{q-1}{q}\right) y(t_0)\lambda(t_0) + \frac{rq}{n}\lambda(t_0).$$

As above in any sufficiently small neighborhood of t_0 as I we get $\frac{d}{dt}\lambda(u, v, t) > \left(\gamma + \frac{q-1}{q}\right)\lambda(u, v, t)y(t)+$ $\lambda(u, v, t)\frac{rq}{n}$, for t_1 sufficiently close to t_0 where $[t_1, t_0] \subseteq I$, and

$$\lambda(u, v, t_0) = \lambda(t_0), \quad \lambda(u, v, t_1) \geq \lambda(t_1). \tag{3.3}$$

Now by integrating from both sides we get

$$\frac{\lambda(t_0)}{\lambda(t_1)} \geq \frac{e^{-\frac{rq}{n}t_1}\left(2R_{min}(0)e^{-\frac{r}{n}t_1} + (r - 2R_{min}(0))\right)^{\frac{n}{2}\left(\gamma+\frac{q-1}{q}\right)}}{e^{-\frac{rq}{n}t_0}\left(2R_{min}(0)e^{-\frac{r}{n}t_0} + (r - 2R_{min}(0))\right)^{\frac{n}{2}\left(\gamma+\frac{q-1}{q}\right)}},$$

and since is arbitrary we get the result. □

Proof of Theorem 2.6. Consider M as a compact Riemannian manifold, we recall the p-Reilly formula as

$$\int_M \left[(\Delta_p u)^2 - |\nabla u|^{2p-4}|Hess\, u|_A^2\right] d\mu = \int_M |\nabla u|^{2p-4} Ric\,(\nabla u, \nabla u)\, d\mu,$$

where u is smooth function and

$$|Hess\, u|_A^2 = |Hess\, u|^2 + \frac{p-2}{2}\frac{|\nabla|\nabla u|^2 u|^2}{|\nabla u|^2} + \frac{p-4}{4}\frac{<\nabla u, \nabla|\nabla u|^2>^2}{|\nabla u|^4}.$$

We also have

$$|\nabla u|^{2p-4}|Hess\, u|_A^2 \geq \frac{1}{n}(\Delta_p u)^2 \geq \frac{1}{n+1}(\Delta_p u)^p, \tag{3.4}$$

and

$$-\Delta_p u = \lambda u|u|^{p-2} + \lambda|u|^\alpha|v|^\beta v,$$

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thus

$$\int_M (\Delta_p u)^2 \, d\mu = \lambda^2 \int_M u^2 |u|^{2p-4} d\mu + \lambda^2 \int_M |u|^{2\alpha} |v|^{2\beta} v^2 d\mu + \lambda^2 \int_M 2u|u|^{p-2}|u|^{\alpha}|v|^{\beta} v d\mu.$$

Now consider

$$I = \int_M u^2|u|^{2p-4} d\mu + \int_M |u|^{2\alpha}|v|^{2\beta} v^2 d\mu + \int_M 2|u|^{p-2}|u|^{\alpha}|v|^{\beta} uv d\mu.$$

Then by the inequality (3.4) we have

$$\int_M |\nabla u|^{2p-4} |Hess\, u|_A^2 d\mu - \frac{1}{n+1}\lambda^2 I \geq 0,$$

which finally implies that

$$\int_M \left[(\Delta_p u)^2 - |\nabla u|^{2p-4} |Heaa\, u|_A^2 \right] d\mu \leq \left(\frac{n}{n+1} \right) \lambda^2 I,$$

where by Reilly formula we get

$$\int_M |\nabla u|^{2p-4} Ric\,(\nabla u, \nabla u)\, d\mu \leq \frac{n}{n+1}\lambda^2 I.$$

Since $R_{ij} - \gamma R g_{ij} \geq 0$ we obtain

$$\begin{aligned} \frac{n}{n+1}\lambda^2 I &\geq \gamma \int_M |\nabla u|^{2p-4} R g_{ij} \nabla_i u \nabla_j u d\mu \\ &= \gamma \int_M R |\nabla u|^{2p-2} d\mu \\ &\geq \gamma R_{min}(t) \int_M |\nabla u|^{2p-2} d\mu. \end{aligned}$$

The manifold M is compact and $\int_M |\nabla u|^{2p-2} d\mu$ is positive; since $R_{min}(t) \longrightarrow \infty$ we have

$$\lambda(t) \longrightarrow +\infty,$$

as $t \to T$. This completes the proof. □

Theorem 2.7, recall that on surfaces we have the following bounds for the scalar curvature R along the normalized Ricci flow

- If $r < 0$ then

$$r - ce^{rt} \leq R \leq r + ce^{rt}.$$

- If $r = 0$ then

$$-\frac{c}{1+ct} \leq R \leq c.$$

- If $r > 0$ then

$$-ce^{rt} \leq R \leq r + ce^{rt}.$$

Where c is real constant, by this classification and $R_{ij} = \frac{1}{2} R g_{ij}$ which holds on 2-surfaces, the proof is based on some calculations.

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Proof of Theorem 2.7. By Proposition 3.2 for $p \geq q \geq 2$ we get

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} &= \lambda(t_0)\int_M R|u|^{\alpha}|v|^{\beta}uv d\mu + \frac{\alpha+1}{p}\lambda(t_0)\int_M R|u|^p d\mu \\ &+ \frac{\beta+1}{q}\lambda(t_0)\int_M R|v|^q d\mu + \left(\frac{p}{2}-1\right)\frac{\alpha+1}{p}\int_M R|\nabla u|^p d\mu \\ &+ \left(\frac{q}{2}-1\right)\frac{\beta+1}{q}\int_M R|\nabla v|^q d\mu + \frac{pr}{n}.\frac{\alpha+1}{p}\int_M |\nabla u|^p d\mu \\ &+ \frac{qr}{n}.\frac{\beta+1}{q}\int_M R|\nabla v|^q d\mu \\ &\geq \left(r-ce^{rt}\right)\lambda(t_0) + \left(\frac{p}{2}-1\right)\left(r-ce^{rt}\right)\lambda(t_0) + \frac{qr}{n}\lambda(t_0).\end{aligned}$$

Now by integrating from both sides on a sufficient neighborhood of t_0 as I we get what we were looking for, the other parts are proved similarly. □

4. 3-homogeneous manifolds and Bianchi classes

Locally homogeneous 3-manifolds have been divided into 9 classes in two groups. The first is contained $H(3)$, $H(2)\times\mathbb{R}^1$ and $SO(3)\times\mathbb{R}^1$, and the other includes $\mathbb{R}^3$, SU(2), SL(2, $\mathbb{R}$), *Heisenberg*, E(1, 1) and E(2), in which the second group are called Bianchi classes.

Also there are some published works in a case of Bianchi classes, for example Hou in [18] has given some bounds for the first eigenvalue of $-\Delta$ in a case of $u > 0$ under the backward Ricci flow, later Razavi and Korouki in [19] studied similar work for the first eigenvalue of $(-\Delta - R)$ under the Ricci flow.

In this section we are going to give evolution and monotonicity for the first eigenvalue of the (p,q)-elliptic quasilinear system (2.1), in a case of Bianchi classes.

Consider g_0 as a given metric in the Bianchi classes. Milnor in [22] introduced the frame $\{X_i\}_{i=1}^3$ in which both the Ricci tensors and metric are diagonalized and this property is preserved by Ricci flow. We consider the metric g as

$$g(t) = A(t)\left(\theta_1\right)^2 + B(t)\left(\theta_2\right)^2 + C(t)\left(\theta_3\right)^2.$$

Remark 4.1. *In homogeneous condition the scalar curvature R is constant, by this fact and also by proposition 3.1 we get*

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} &= (\alpha+1)\int_M Ric\left(\nabla u,\nabla u\right)|\nabla u|^{p-2}d\mu \\ &+ (\beta+1)\int_M Ric\left(\nabla v,\nabla v\right)|\nabla v|^{q-2}d\mu.\end{aligned} \tag{4.1}$$

Now we are going to give monotonicity for the first eigenvalue of (p,q)-elliptic quasilinear system (2.1) in a case of Bianchi classes separately, where $p \geq q$

Case 1: $\mathbb{R}^3$

In this case all metrics are flat, so for all $t \geq 0$ we have $g(t) = g_0$ where g_0 is constant, therefore $\lambda(t)$ is constant.

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Case 2: Heisenberg

This class is isomorphic to the set of upper-triangular 3×3 matrices endowed with the usual matrix multiplication. Under the metric g_0 we choose a frame $\{X_i\}_{i=1}^3$ in which

$$[X_2, X_3] = X_1, \quad [X_3, X_1] = 0, \quad [X_1, X_2] = 0,$$

also under the normalization $A_0B_0C_0 = 1$ we have

$$R = -\frac{1}{2}A^2, \quad R_{11} = \frac{1}{2}A^3, \quad R_{22} = -\frac{1}{2}A^2B, \quad R_{33} = -\frac{1}{2}A^2C, \quad ||Ric||^2 = \frac{3}{4}A^4,$$

by substituting Ric into the formula (4.1) under the Ricci flow (1.1) we have

$$\begin{aligned} \frac{d}{dt}\lambda(u,v,t)|_{t=t_0} = {} & p\frac{(\alpha+1)}{p}\int_M \Big[\frac{1}{2}A^3\nabla_1 u\nabla_1 u - \frac{1}{2}A^2B\nabla_2 u\nabla_2 u \\ & - \frac{1}{2}A^2C\nabla_3 u\nabla_3 u\Big]|\nabla u|^{p-2}d\mu \\ & + q\frac{(\beta+1)}{q}\int_M \Big[\frac{1}{2}A^3\nabla_1 v\nabla_1 v - \frac{1}{2}A^2B\nabla_2 v\nabla_2 v \\ & - \frac{1}{2}A^2C\nabla_3 v\nabla_3 v\Big]|\nabla v|^{q-2}d\mu \\ & \geq -\frac{p}{2}A^2\lambda(t_0), \end{aligned} \tag{4.2}$$

where by similar process we get

$$\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} \leq \frac{p}{2}A^2\lambda(t_0).$$

Now by integrating from both sides of the bounds of (4.2) in a sufficient neighborhood of t_0 as I we get

$$\lambda(t)e^{\frac{p}{2}\int_0^t A^2 dt},$$

is non-decreasing and also

$$\lambda(t)e^{-\frac{p}{2}\int_0^t A^2 dt},$$

is non-increasing along the Ricci flow (1.1).

Case 3: E(2)

E(2) is the group of isometries of Euclidian plane. In this case we have an Einstein metric and Ricci flow converges exponentially to flat metrics. Dependent to the metric g_0 we choose the frame $\{X_i\}_{i=0}^3$ such that

$$[X_2, X_3] = X_1, \quad [X_3, X_1] = X_2, \quad [X_1, X_2] = 0,$$

In this case under the normalization $A_0B_0C_0 = 1$ we have

$$\begin{aligned} R = -\frac{1}{2}(A-B)^2, \quad R_{11} = \frac{1}{2}A\left(A^2 - B^2\right), \quad R_{22} = \frac{1}{2}B\left(B^2 - A^2\right), \\ R_{33} = -\frac{1}{2}C(A-B)^2, \quad ||Ric||^2 = \frac{1}{2}\left(A^2 - B^2\right)^2 + \frac{1}{4}(A-B)^4, \end{aligned}$$

similar to the case of $Heisenberg$ under Ricci flow (1.1), if we assume $A_0 \geq B_0$ then

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} &= p\frac{(\alpha+1)}{p}\int_M \Big[\frac{1}{2}A\left(A^2-B^2\right)\nabla_1 u\nabla_1 u + \frac{1}{2}B\left(B^2-A^2\right)\nabla_2 u\nabla_2 u \\ &- \frac{1}{2}C\left(A-B\right)^2\nabla_3 u\nabla_3 u\Big]|\nabla u|^{p-2}d\mu \\ &+ q\frac{(\beta+1)}{q}\int_M \Big[\frac{1}{2}A\left(A^2-B^2\right)\nabla_1 v\nabla_1 v + \frac{1}{2}B\left(B^2-A^2\right)\nabla_2 v\nabla_2 v \\ &- \frac{1}{2}C\left(A-B\right)^2\nabla_3 v\nabla_3 v\Big]|\nabla v|^{q-2}d\mu \\ &\geq -\frac{p}{2}\left(A^2-B^2\right)\lambda(t_0).\end{aligned}$$

In a similar way

$$\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} \leq \frac{p}{2}\left(A^2-B^2\right)\lambda(t_0).$$

Similarly by substituting Ric into the evolution formula (4.1) and also by integrating from both sides of this formula in a sufficient neighborhood of t_0 as I we also have

$$\lambda(t)e^{\frac{p}{2}\int_0^t\left(A^2-B^2\right)dt},$$

is non-decreasing and also

$$\lambda(t)e^{-\frac{p}{2}\int_0^t\left(A^2-B^2\right)dt},$$

is non-increasing along the Ricci flow (1.1).

Case 4: E(1,1)

E(1,1) is the group of isometries of the plane with flat Lorentz metric, there is no Einstein metric here and Ricci flow fails to converge, they all are asymptotically cigar degeneracies. For a given metric g_0 similarly by a frame $\{X_i\}_{i=0}^3$ we have

$$[X_1, X_2] = 0, \quad [X_2, X_3] = -X_1, \quad [X_3, X_1] = X_2.$$

Also under the normalization $A_0B_0C_0 = 1$ we have

$$\begin{aligned}&R = -\frac{1}{2}\left(A+B\right)^2, \quad R_{11} = \frac{1}{2}A\left(A^2-B^2\right), \quad R_{22} = \frac{1}{2}B\left(B^2-A^2\right), \\ &R_{33} = -\frac{1}{2}C\left(A+B\right)^2, \quad ||Ric||^2 = \frac{3}{4}A^4.\end{aligned}$$

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By the substituting Ric into the formula (4.1), under the Ricci flow we have

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} &= p\frac{(\alpha+1)}{p}\int_M \Big[\frac{1}{2}A\left(A^2-B^2\right)\nabla_1 u\nabla_1 u+\frac{1}{2}B\left(B^2-A^2\right)\nabla_2 u\nabla_2 u\\ &-\frac{1}{2}C\left(A+B\right)^2\nabla_3 u\nabla_3 u\Big]|\nabla u|^{p-2}d\mu\\ &+q\frac{(\beta+1)}{q}\int_M \Big[\frac{1}{2}A\left(A^2-B^2\right)\nabla_1 v\nabla_1 v+\frac{1}{2}B\left(B^2-A^2\right)\nabla_2 v\nabla_2 v\\ &-\frac{1}{2}C\left(A+B\right)^2\nabla_3 v\nabla_3 v\Big]|\nabla v|^{q-2}d\mu\\ &\geq -\frac{p}{2}\left(A+B\right)^2\lambda(t_0),\end{aligned}$$

by the similar process

$$\frac{d}{dt}\lambda(u,v,t)|_{t=t_0}\leq \frac{p}{2}\left(A+B\right)^2\lambda(t_0).$$

This implies that by integrating from both sides in a sufficient neighborhood of t_0 as I we conclude that

$$\lambda(t)e^{\frac{p}{2}\int_0^t (A+B)^2 dt},$$

is non-decreasing along and

$$\lambda(t)e^{-\frac{p}{2}\int_0^t (A+B)^2 dt}$$

is non-increasing along the Ricci flow (1.1).

Case 5: SU(2)

Similarly in this class we have Einstein metrics and Ricci flow converges exponentially into these metrics, also by the frame$\{X_i\}_{i=0}^3$ we have

$$[X_2,X_3]=X_1,\quad [X_3,X_1]=X_2,\quad [X_1,X_2]=X_3,$$

In this case under the normalization $A_0B_0C_0=1$, we have

$$\begin{aligned}&R=\eta A^2,\quad R_{11}=\frac{1}{2}A[A^2-(B-C)^2],\\ &R_{22}=\frac{1}{2}B[B^2-(A-C)^2],\quad R_{33}=\frac{1}{2}C[C^2-(A-B)^2],\\ &||Ric||^2=\frac{1}{4}\{[A^2-(B-C)^2]^2+[B^2-(A-C)^2]^2+[C^2-(A-B)^2]^2\},\end{aligned}$$

where

$$\eta=\frac{1}{2}\Big\{1-\Big(\frac{B_0}{A_0}-\frac{C_0}{A_0}\Big)^2+\Big(\frac{B_0}{A_0}\Big)^2-\Big(1-\frac{C_0}{A_0}\Big)^2+\Big(\frac{C_0}{A_0}\Big)^2-\Big(1-\frac{B_0}{A_0}\Big)^2\Big\}.$$

In this case by Cao [6] we get

- If $A_0=B_0=C_0$ then $\lambda(t)$ is constant.

- If $A_0 = B_0 > C_0$ then by similar calculation as above we have
$$\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} \geq -\frac{p}{2}\Big[A^2 - (A-C)^2\Big]\lambda(t_0),$$
and also
$$\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} \leq \frac{p}{2}A^2\lambda(t_0).$$

This implies that by integrating from both sides in a sufficient neighborhood of t_0 as I we conclude that
$$\lambda(t)e^{\frac{p}{2}\int_0^t [A^2-(A-C)^2]dt},$$
is non-decreasing and also
$$\lambda(t)e^{-\frac{p}{2}\int_0^t A^2 dt}$$
is non-increasing along the Ricci flow (1.1).

Case 6: SL(2,$\mathbb{R}$)

There is no Einstein metrics here and the Ricci flow doesn't converge and develops a pancake degeneracy, also by the frame $\{X_i\}_{i=0}^3$, we get
$$[X_2, X_3] = -X_1, \quad [X_3, X_1] = X_2, \quad [X_1, X_2] = X_3,$$
in this case we also have
$$R = \eta A^2, \quad R_{11} = \frac{1}{2}A[A^2 - (B-C)^2], \quad R_{22} = \frac{1}{2}B[B^2 - (A+C)^2]$$
$$R_{33} = \frac{1}{2}C[C^2 - (A+B)^2],$$
$$||Ric||^2 = \frac{1}{4}\big\{[A^2 - (B-C)^2]^2 + [B^2 - (A+C)^2]^2 + [C^2 - (A+B)^2]^2\big\},$$
in which
$$\eta = -\frac{1}{2}\Big\{1 + \Big(\frac{B_0}{A_0}\Big)^2 + \Big(\frac{C_0}{A_0}\Big)^2 + 2\frac{B_0}{A_0} + 2\frac{C_0}{A_0} - 2\frac{B_0}{A_0}\frac{C_0}{A_0}\Big\}.$$

In this case by simple calculation and by Cao [5,6], and also by substituting Ric into the formula (4.1) we find that

- If $A > B = C$ then
$$\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} \geq \frac{p}{2}[B^2 - (A+B)^2]\lambda(t_0),$$
and
$$\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} \leq \frac{p}{2}A^2\lambda(t_0),$$
which by integrating in a sufficient neighborhood of t_0 as I, implies
$$\lambda(t)e^{\frac{p}{2}\int_0^t [B^2-(A+B)^2]dt},$$

is non-decreasing and also

$$\lambda(t)e^{-\frac{p}{2}\int_0^t A^2 dt},$$

is non-increasing along the Ricci flow (1.1). Also in a similar way we get

- If $A \leq B - C$ we find

$$\lambda(t)e^{-\frac{p}{2}\int_0^t [C^2-(A+B)^2]dt},$$

is non-decreasing and

$$\lambda(t)e^{-\frac{p}{2}\int_0^t [B^2-(A+C)^2]dt},$$

is non-increasing along the Ricci flow (1.1).

5. Einstein metrics; an important example

One of the most interesting case of such metrics are Einstein metrics in which *Ric* tensor is scalar coefficient of given metric. In this case we are going to give an example of monotonicity of the first eigenvalue of (p, q)-elliptic quasilinear system (2.1) in a case of Einstein metrics.

Example 5.1. *Consider (M^n, g_0) as an Einstein manifold, then there exist a constant a such that $Ric(g_0) = ag_0$. The solution to the Ricci flow on M is*

$$g(t) = (1-2at)\, g_0,$$

therefore it implies that

$$Ric\,(g(t)) = \frac{a}{1-2at} g(t) \qquad R_{g(t)} = \frac{an}{1-2at}.$$

In this case we have

$$\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} = \frac{a}{1-2at}\Big[p\frac{\alpha+1}{p}\int_M |\nabla u|^p d\mu + q\frac{\beta+1}{q}\int_M |\nabla v|^q d\mu\Big],$$

by the assumption $p \geq q$ we find that

$$\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} \geq \frac{qa}{1-2at}\lambda(t_0).$$

Now by integrating from both sides in a sufficient neighborhood as $I = [t_1, t_0]$ and also under similar assumption as theorems proved before, we get

$$\lambda(t)\,(1-2at)^{\frac{q}{2}},$$

is increasing along the Ricci flow (1.1).

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References

[1] A. Abolarinwa, Evolution and monotonicity of the first eigenvalue of p-Laplacian under the Ricci harmonic flow, *J. Appl. Anal.* **21** (2015), no. 2, 147–160.

[2] S. Azami, Eigenvalues of a (p, q)-Laplacian system under the mean curvature flow, *Iran. J. Sci. Technol. Trans. A Sci.* **43** (2019), no. 4, 1881–1888.

[3] K. Brown and Y. Zhang, On a system of reaction-diffusion equations describing a population with two age groups, *J. Math. Anal. Appl.* **282** (2003), no. 2, 444–452.

[4] X. Cao, Eigenvalues of $(-\Delta + \frac{R}{2})$ on manifolds with nonnegative curvature operator, *Math. Ann.* **337** (2007), no. 2, 435–441.

[5] X. Cao, J. Guckenheimer and L. Saloff-Coste, The backward behavior of the Ricci and cross curvature flows on $SL(2, \mathbb{R})$, *Comm. Anal. Geom.* **17** (2009), no. 4, 777–796.

[6] X. Cao and L. Saloff-Coste, Backward Ricci flow on locally homogeneous 3-manifolds, *Comm. Anal. Geom.* **17** (2009), no. 2, 305–325.

[7] S. Cheng, Eigenfunctions and eigenvalues of Laplacian, Proc. Sympos. Pure Math. 27 part 2, Differential Geometry, (eds. S. Chern and R. Osserman), 185–193, Amer. Math. Soc., Providence, R.I., 1975.

[8] Q. Cheng and H. Yang, Estimates on eigenvalues of Laplacian, *Math. Ann.* **331** (2005), no. 2, 445–460.

[9] Y. Chen, Y. Giga and S. Goto, Uniqueness ans existence of viscosity solutions of generalized mean curvature flow equations, *J. Differential Geom.* **33** (1991) 749–786.

[10] Y. Choi, Z. Huan and R. Lui, Global existence of solutions of a strongly coupled quasilinear parabolic system with application to electrochemistry, *J. Differential Equations* **194** (2003), no. 2, 406–432.

[11] B. Chow and D. Knopf, The Ricci Flow: An Introduction, American Mathematical Society, Providence, RI, 2004.

[12] A. Constantin, J. Escher and Z. Yin, Global solutions for quasilinear parabolic systems, *J. Differential Equations* **197** (2004), no. 1, 73–84.

[13] E. Dancer and Y. Du, Effects of certain degeneracies in the predator-prey model, *SIAM J. Math. Anal.* **34** (2002), no. 2, 292–314.

[14] D. De Turk, Deforming metrics in the direction of their Ricci tensors, *J. Differential Geom.* **18** (1983), no. 1, 157–162.

[15] J. Eells and J. Sampson, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.* **86** (1964), 109–160.

[16] D. Friedan, Nonlinear models in $2 + \epsilon$ dimensions, *Ann. Physics* **163** (1985), no. 2, 318–419.

[17] R. Hamilton, Three-manifolds with positive Ricci curvature, *J. Differential Geometry* **17** (1982), no. 2, 255–306.

[18] S. Hou, Eigenvalues under the backward Ricci flow on locally homogeneous closed 3-manifolds, *Acta Math. Sin. (Engl. Ser.)* **34** (2018), no. 7, 1179–1194.

[19] F. Korouki and A. Razavi, Bounds for the first eigenvalue of $(-\Delta - R)$ under the Ricci flow on Bianchi classes, *Bull. Braz. Math. Soc. (N.S.)* **51** (2020), no. 2, 641–651.

[20] Y. Li, Eigenvalues and entropies under the harmonic-Ricci flow, *Pacific J. Math.* **267** (2014), no. 1, 141–184.

[21] L. Ma, Eigenvalue monotonicity for the Ricci-Hamilton flow, *Ann. Global Anal. Geom.* **29** (2006), no. 3, 287–292.

[22] J. Milnor, Curvatures of left invariant metrics on Lie groups, *Adv. Math.* **21** (1976), no. 3, 293–329.

[23] G. Perelman, The entropy formula for the Ricci flow and it's geometric applications, Arxiv (2002).

[24] B. Santoro, Introduction to evolution equations in geometry, IMPA, Estrada Doa Castorina, 110 22460-320, Rio de Janeiro RJ (2009).

[25] Y. Z. Wang, Gradient estimates on the weighted p-Laplace heat equation, *J. Differ. Equ.* **264** (2018), 506–524.

[26] J. Wu, E. Wang and Y. Zheng, First eigenvalue of p-Laplace operator along the Ricci flow, *Ann. Glob. Anal. Geom.* **38** (2010), 27–55.

[27] N. Zographopoulos, On the principal eigenvalue of degenerate quasilinear elliptic systems, *Math. Nachr.* **281** (2008), no. 9, 1351–1365.

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