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GENERALIZED TRAPEZOID TYPE INEQUALITIES FOR FUNCTIONS WITH VALUES IN BANACH SPACES

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ABSTRACT. Let E be a complex Banach space. In this paper we show among others that, if $\alpha : [a,b] \to \mathbb{C}$ is continuous and $Y : [a,b] \to E$ is strongly differentiable on the interval (a,b), then for all $u \in [a,b]$,

$$\left\| \left(\int_u^b \alpha(s)\,ds \right) Y(b) + \left(\int_a^u \alpha(s)\,ds \right) Y(a) - \int_a^b \alpha(t) Y(t)\,dt \right\|$$
$$\leq \begin{cases} \max\left\{ \int_u^b |\alpha(s)|\,ds, \int_a^u |\alpha(s)|\,ds \right\} \int_a^b \|Y'(t)\|\,dt, \\ \\ \left[\int_u^b (b-t)|\alpha(t)|\,dt + \int_a^u (t-a)|\alpha(t)|\,dt \right] \sup_{t\in[a,b]} \|Y'(t)\|, \\ \\ \leq (b-a)^{1/p} \left[\left(\int_u^b |\alpha(s)|\,ds \right)^p + \left(\int_a^u |\alpha(s)|\,ds \right)^p \right]^{1/p} \\ \times \left(\int_a^b \|Y'(t)\|^q\,dt \right)^{1/q} \end{cases}$$

for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Applications for operator monotone functions with examples for power and logarithmic functions are also given.

1. Introduction

In 2001, Dragomir et al. [11] obtained the following generalized trapezoid inequality:

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Theorem 1.1. *If $g:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$, then*

$$\left|\int_a^b f(t)\,g(t)\,dt-f(b)\int_u^b g(t)\,dt-f(a)\int_a^u g(t)\,dt\right| \le\left[\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right]\sup_{t\in[a,b]}|g(t)|\bigvee_a^b(f) \tag{1.1}$$

for all $u\in[a,b]$, where $\bigvee_a^b(f)$ is the total variation of f on $[a,b]$.

In particular, we have the mid-point trapezoid inequality

$$\left|\int_a^b f(t)\,g(t)\,dt-f(b)\int_{\frac{a+b}{2}}^b g(t)\,dt-f(a)\int_a^{\frac{a+b}{2}} g(t)\,dt\right| \le\frac{1}{2}(b-a)\sup_{t\in[a,b]}|g(t)|\bigvee_a^b(f). \tag{1.2}$$

The constant $1/2$ is sharp in the sense that it cannot be replaced by a smaller quantity.

Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space. The continuous function f is operator convex on the interval I if for any selfadjoint operators A and B with spectra $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset I$,

$$f((1-t)A+tB)\le(1-t)f(A)+tf(B)$$

for all $t\in[0,1]$.

In the recent paper [10] we obtained the following result for functions in Hilbert spaces H:

Theorem 1.2. *Let f be an operator convex function on I and A, B, $A\ne B$, selfadjoint operators on H with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset I$. If f is Gâteaux differentiable on $[A,B]:=\{(1-t)A+tB,\ t\in[0,1]\}$ and $p:[0,1]\to[0,\infty)$ is Lebesgue integrable and symmetric, namely $p(1-t)=p(t)$ for all $t\in[0,1]$, then*

$$0\le\left(\int_0^1 p(t)\,dt\right)\frac{f(A)+f(B)}{2}-\int_0^1 p(t)\,f((1-t)A+tB)\,dt \le\frac{1}{2}\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt\left[\nabla f_B(B-A)-\nabla f_A(B-A)\right], \tag{1.3}$$

where $\nabla f_C(V)$ is the Gâteaux derivative in C over the direction V.

In particular, for $p\equiv1$ we get

$$0\le\frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt \le\frac{1}{8}\left[\nabla f_B(B-A)-\nabla f_A(B-A)\right]. \tag{1.4}$$

For some trapezoid operator inequalities in Hilbert spaces, see [6, 8, 9] and [7].

Let E be a complex Banach space. We say that the vector valued function $Y:[a,b]\to E$ is *strongly differentiable* on the interval (a,b) if the limit

$$Y'(t):=\lim_{h\to0}\frac{1}{h}\left[Y(t+h)-Y(t)\right]$$

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exists in the norm topology for all $t \in (a, b)$.

In this paper we show among others that, if $\alpha : [a, b] \to \mathbb{C}$ is continuous and $Y : [a, b] \to E$ is strongly differentiable on the interval (a, b), then for all $u \in [a, b]$,

$$\left\| \left(\int_u^b \alpha(s)\, ds \right) Y(b) + \left(\int_a^u \alpha(s)\, ds \right) Y(a) - \int_a^b \alpha(t) Y(t)\, dt \right\|$$
$$\leq \begin{cases} \max\left\{ \int_u^b |\alpha(s)|\, ds, \int_a^u |\alpha(s)|\, ds \right\} \int_a^b \|Y'(t)\|\, dt, \\ \\ \left[\int_u^b (b-t) |\alpha(t)|\, dt + \int_a^u (t-a) |\alpha(t)|\, dt \right] \sup_{t \in [a,b]} \|Y'(t)\|, \\ \\ \leq (b-a)^{1/p} \left[\left(\int_u^b |\alpha(s)|\, ds \right)^p + \left(\int_a^u |\alpha(s)|\, ds \right)^p \right]^{1/p} \\ \times \left(\int_a^b \|Y'(t)\|^q\, dt \right)^{1/q} \end{cases}$$

for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Applications for operator monotone functions on Hilbert spaces with examples for power and logarithmic functions are also given.

2. General trapezoid inequalities

We have the following weighted version of generalized trapezoid inequality for two functions with values in Banach spaces:

Theorem 2.1. *Assume that $\alpha : [a, b] \to \mathbb{C}$, $Y : [a, b] \to E$ are continuous and Y is strongly differentiable on (a, b), then for all $u \in [a, b]$*

$$\left\| \left(\int_u^b \alpha(s)\, ds \right) Y(b) + \left(\int_a^u \alpha(s)\, ds \right) Y(a) - \int_a^b \alpha(t) Y(t)\, dt \right\| \leq C(\alpha, Y, u), \tag{2.1}$$

where

$$C(\alpha, Y, u) := \int_u^b \left(\int_u^t |\alpha(s)|\, ds \right) \left\| Y'(t) \right\| dt + \int_a^u \left(\int_t^u |\alpha(s)|\, ds \right) \left\| Y'(t) \right\| dt.$$

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We also have the bounds

$$C(\alpha, Y, u) \tag{2.2}$$

$$\leq \begin{cases} \int_u^b |\alpha(s)|\, ds \int_u^b \|Y'(t)\|\, dt, \\ \\ \left[\int_u^b \left(\int_u^t |\alpha(s)|\, ds\right)^p dt\right]^{1/p} \left(\int_u^b \|Y'(t)\|^q\, dt\right)^{1/q}, \\ \\ \int_u^b \left(\int_u^t |\alpha(s)|\, ds\right) dt \sup_{t\in[u,b]} \|Y'(t)\|, \end{cases}$$

$$+ \begin{cases} \left(\int_a^u |\alpha(s)|\, ds\right) \int_a^u \|Y'(t)\|\, dt, \\ \\ \left[\int_a^u \left(\int_t^u |\alpha(s)|\, ds\right)^p dt\right]^{1/p} \left(\int_a^u \|Y'(t)\|^q\, dt\right)^{1/q}, \\ \\ \int_a^u \left(\int_t^u |\alpha(s)|\, ds\right) dt \sup_{t\in[a,u]} \|Y'(t)\|, \end{cases}$$

where p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $u \in [a, b]$. Using the integration by parts formula for Bochner integral, we have

$$\begin{aligned} &\int_a^b \left(\int_a^t \alpha(s)\, ds - \int_a^u \alpha(s)\, ds\right) Y'(t)\, dt \\ &= \left(\int_a^t \alpha(s)\, ds - \int_a^u \alpha(s)\, ds\right) Y(t)\Big|_a^b \\ &- \int_a^b \left(\int_a^t \alpha(s)\, ds - \int_a^u \alpha(s)\, ds\right)' Y(t)\, dt \\ &= \left(\int_a^b \alpha(s)\, ds - \int_a^u \alpha(s)\, ds\right) Y(b) \\ &- \left(\int_a^a \alpha(s)\, ds - \int_a^u \alpha(s)\, ds\right) Y(a) \\ &- \int_a^b \left(\int_a^t \alpha(s)\, ds - \int_a^u \alpha(s)\, ds\right)' Y(t)\, dt \\ &= \left(\int_u^b \alpha(s)\, ds\right) Y(b) + \left(\int_a^u \alpha(s)\, ds\right) Y(a) - \int_a^b \alpha(t)\, Y(t)\, dt. \end{aligned} \tag{2.3}$$

Also,

$$\begin{aligned} &\int_a^b \left(\int_a^t \alpha(s)\, ds - \int_a^u \alpha(s)\, ds\right) Y'(t)\, dt \\ &= \int_a^u \left(\int_a^t \alpha(s)\, ds - \int_a^u \alpha(s)\, ds\right) Y'(t)\, dt \\ &+ \int_u^b \left(\int_a^t \alpha(s)\, ds - \int_a^u \alpha(s)\, ds\right) Y'(t)\, dt \\ &= -\int_a^u \left(\int_t^u \alpha(s)\, ds\right) Y'(t)\, dt + \int_u^b \left(\int_u^t \alpha(s)\, ds\right) Y'(t)\, dt. \end{aligned} \tag{2.4}$$

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By utilising (2.3) and (2.4) we derive the following identity of interest

$$\left(\int_u^b \alpha(s)\,ds\right) Y(b) + \left(\int_a^u \alpha(s)\,ds\right) Y(a) - \int_a^b \alpha(t) Y(t)\,dt \tag{2.5}$$
$$= \int_u^b \left(\int_u^t \alpha(s)\,ds\right) Y'(t)\,dt - \int_a^u \left(\int_t^u \alpha(s)\,ds\right) Y'(t)\,dt$$

for all $u \in [a,b]$.

Taking the norm in (2.5) and using the properties of the integral, we get

$$\begin{aligned}
&\left\| \left(\int_u^b \alpha(s)\,ds\right) Y(b) + \left(\int_a^u \alpha(s)\,ds\right) Y(a) - \int_a^b \alpha(t) Y(t)\,dt \right\| \\
&\leq \left\| \int_u^b \left(\int_u^t \alpha(s)\,ds\right) Y'(t)\,dt \right\| + \left\| \int_a^u \left(\int_t^u \alpha(s)\,ds\right) Y'(t)\,dt \right\| \\
&\leq \int_u^b \left\| \left(\int_u^t \alpha(s)\,ds\right) Y'(t) \right\| dt + \int_a^u \left\| \left(\int_t^u \alpha(s)\,ds\right) Y'(t) \right\| dt \\
&\leq \int_u^b \left| \int_u^t \alpha(s)\,ds \right| \left\| Y'(t) \right\| dt + \int_a^u \left| \int_t^u \alpha(s)\,ds \right| \left\| Y'(t) \right\| dt \\
&\leq \int_u^b \left(\int_u^t |\alpha(s)|\,ds\right) \left\| Y'(t) \right\| dt + \int_a^u \left(\int_t^u |\alpha(s)|\,ds\right) \left\| Y'(t) \right\| dt \\
&= C(\alpha, Y, u),
\end{aligned} \tag{2.6}$$

which proves (2.1).

Using Hölder's inequality, we get for $p,\ q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, that

$$\begin{aligned}
&\int_u^b \left(\int_u^t |\alpha(s)|\,ds\right) \left\| Y'(t) \right\| dt \\
&\leq \begin{cases}
\sup_{t\in[u,b]} \left(\int_u^t |\alpha(s)|\,ds\right) \int_u^b \|Y'(t)\|\,dt, \\[2ex]
\left[\int_u^b \left(\int_u^t |\alpha(s)|\,ds\right)^p dt\right]^{1/p} \left(\int_u^b \|Y'(t)\|^q\,dt\right)^{1/q}, \\[2ex]
\int_u^b \left(\int_u^t |\alpha(s)|\,ds\right) dt \sup_{t\in[u,b]} \|Y'(t)\|,
\end{cases} \\
&= \begin{cases}
\left(\int_u^b |\alpha(s)|\,ds\right) \int_u^b \|Y'(t)\|\,dt, \\[2ex]
\left[\int_u^b \left(\int_u^t |\alpha(s)|\,ds\right)^p dt\right]^{1/p} \left(\int_u^b \|Y'(t)\|^q\,dt\right)^{1/q}, \\[2ex]
\int_u^b \left(\int_u^t |\alpha(s)|\,ds\right) dt \sup_{t\in[u,b]} \|Y'(t)\|,
\end{cases}
\end{aligned}$$

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and

$$\begin{aligned}
&\int_a^u \left(\int_t^u |\alpha(s)|\,ds\right) \left\|Y'(t)\right\| dt \\
&\leq \begin{cases} \sup_{t\in[a,u]} \left(\int_t^u |\alpha(s)|\,ds\right) \int_a^u \|Y'(t)\|\,dt, \\ \\ \left[\int_a^u \left(\int_t^u |\alpha(s)|\,ds\right)^p dt\right]^{1/p} \left(\int_a^u \|Y'(t)\|^q\,dt\right)^{1/q}, \\ \\ \int_a^u \left(\int_t^u |\alpha(s)|\,ds\right) dt \sup_{t\in[a,u]} \|Y'(t)\|, \end{cases} \\
&= \begin{cases} \left(\int_a^u |\alpha(s)|\,ds\right) \int_a^u \|Y'(t)\|\,dt, \\ \\ \left[\int_a^u \left(\int_t^u |\alpha(s)|\,ds\right)^p dt\right]^{1/p} \left(\int_a^u \|Y'(t)\|^q\,dt\right)^{1/q}, \\ \\ \int_a^u \left(\int_t^u |\alpha(s)|\,ds\right) dt \sup_{t\in[a,u]} \|Y'(t)\|. \end{cases}
\end{aligned}$$

By making use of (2.6) we deduce (2.2). □

Corollary 2.2. *With the assumptions of Theorem 2.1, we have*

$$\begin{aligned}
&\left\|\left(\int_u^b \alpha(s)\,ds\right) Y(b) + \left(\int_a^u \alpha(s)\,ds\right) Y(a) - \int_a^b \alpha(t) Y(t)\,dt\right\| \\
&\leq \int_u^b |\alpha(s)|\,ds \int_u^b \left\|Y'(t)\right\| dt + \int_a^u |\alpha(s)|\,ds \int_a^u \left\|Y'(t)\right\| dt \\
&\leq \begin{cases} \max\left\{\int_u^b |\alpha(s)|\,ds, \int_a^u |\alpha(s)|\,ds\right\} \int_a^b \|Y'(t)\|\,dt, \\ \\ \int_a^b |\alpha(s)|\,ds \max\left\{\int_u^b \|Y'(t)\|\,dt, \int_a^u \|Y'(t)\|\,dt\right\}, \end{cases} \\
&\leq \int_a^b |\alpha(s)|\,ds \int_a^b \left\|Y'(t)\right\| dt,
\end{aligned} \tag{2.7}$$

for all $u \in [a, b]$.

The proof follows by the first branches in the bounds (2.2).

Remark 2.3. *If* $m \in (a, b)$ *is such that*

$$\int_a^m |\alpha(s)|\,ds = \int_m^b |\alpha(s)|\,ds = \frac{1}{2}\int_a^b |\alpha(s)|\,ds, \tag{2.8}$$

then by (2.6) *we get*

$$\begin{aligned}
&\left\|\left(\int_m^b \alpha(s)\,ds\right) Y(b) + \left(\int_a^m \alpha(s)\,ds\right) Y(a) - \int_a^b \alpha(t) Y(t)\,dt\right\| \\
&\leq \frac{1}{2}\int_a^b |\alpha(s)|\,ds \int_a^b \left\|Y'(t)\right\| dt.
\end{aligned} \tag{2.9}$$

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Corollary 2.4. *With the assumptions of Theorem 2.1, we have*

$$\begin{aligned}\text{(2.10)}\qquad & \left\| \left(\int_u^b \alpha(s)\,ds \right) Y(b) + \left(\int_a^u \alpha(s)\,ds \right) Y(a) - \int_a^b \alpha(t) Y(t)\,dt \right\| \\ & \le \left[\int_u^b (b-t)\left|\alpha(t)\right| dt + \int_a^u (t-a)\left|\alpha(t)\right| dt \right] \sup_{t\in[a,b]} \left\| Y'(t) \right\|\end{aligned}$$

for all $u\in[a,b]$.

Proof. From the third branches in the bounds in (2.2) we have

$$\begin{aligned}\text{(2.11)}\qquad & \left\| \left(\int_u^b \alpha(s)\,ds \right) Y(b) + \left(\int_a^u \alpha(s)\,ds \right) Y(a) - \int_a^b \alpha(t) Y(t)\,dt \right\| \\ & \le \int_u^b \left(\int_u^t |\alpha(s)|\,ds \right) dt \sup_{t\in[u,b]} \left\| Y'(t) \right\| \\ & + \int_a^u \left(\int_t^u |\alpha(s)|\,ds \right) dt \sup_{t\in[a,u]} \left\| Y'(t) \right\| \\ & \le \sup_{t\in[a,b]} \left\| Y'(t) \right\| \left[\int_u^b \left(\int_u^t |\alpha(s)|\,ds \right) dt + \int_a^u \left(\int_t^u |\alpha(s)|\,ds \right) dt \right].\end{aligned}$$

Using integration by parts, we have for $u\in[a,b]$ that

$$\begin{aligned}\int_u^b \left(\int_u^t |\alpha(s)|\,ds \right) dt &= \left(\int_u^t |\alpha(s)|\,ds \right) t \Big|_u^b - \int_u^b t\,|\alpha(t)|\,dt \\ &= \left(\int_u^b |\alpha(s)|\,ds \right) b - \int_u^b t\,|\alpha(t)|\,dt \\ &= \int_u^b (b-t)\,|\alpha(t)|\,dt\end{aligned}$$

and

$$\begin{aligned}\int_a^u \left(\int_t^u |\alpha(s)|\,ds \right) dt &= \left(\int_t^u |\alpha(s)|\,ds \right) t \Big|_a^u + \int_a^u t\,|\alpha(t)|\,dt \\ &= -\left(\int_a^u |\alpha(s)|\,ds \right) a + \int_a^u t\,|\alpha(t)|\,dt \\ &= \int_a^u (t-a)\,|\alpha(t)|\,dt,\end{aligned}$$

which, by (2.11) provides the desired result (2.10). □

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Remark 2.5. *By making use of Hölder's integral inequality, we have for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ that*

$$\int_u^b (b-t)\left|\alpha(t)\right| dt \leq \begin{cases} \sup_{t\in[u,b]}(b-t)\int_u^b \left|\alpha(t)\right| dt, \\ \\ \left(\int_u^b (b-t)^q\,dt\right)^{1/q}\left(\int_u^b \left|\alpha(t)\right|^p dt\right)^{1/p}, \\ \\ \int_u^b (b-t)\,dt\sup_{t\in[u,b]}\left|\alpha(t)\right|, \end{cases}$$
$$= \begin{cases} (b-u)\int_u^b \left|\alpha(t)\right| dt, \\ \\ \frac{(b-u)^{1+1/q}}{(q+1)^{1/q}}\left(\int_u^b \left|\alpha(t)\right|^p dt\right)^{1/p}, \\ \\ \frac{1}{2}(b-u)^2\sup_{t\in[u,b]}\left|\alpha(t)\right| \end{cases}$$

and

$$\int_a^u (t-a)\left|\alpha(t)\right| dt \leq \begin{cases} \sup_{t\in[a,u]}(t-a)\int_a^u \left|\alpha(t)\right| dt, \\ \\ \left(\int_a^u (t-a)^q\,dt\right)^{1/q}\left(\int_a^u \left|\alpha(t)\right|^p dt\right)^{1/p}, \\ \\ \int_a^u (t-a)\,dt\sup_{t\in[a,u]}\left|\alpha(t)\right|, \end{cases}$$
$$= \begin{cases} (u-a)\int_a^u \left|\alpha(t)\right| dt, \\ \\ \frac{(u-a)^{1+1/q}}{(q+1)^{1/q}}\left(\int_a^u \left|\alpha(t)\right|^p dt\right)^{1/p}, \\ \\ \frac{1}{2}(u-a)^2\sup_{t\in[a,u]}\left|\alpha(t)\right|. \end{cases}$$

By (2.10) we then get

$$\left\|\left(\int_u^b \alpha(s)\,ds\right)Y(b)+\left(\int_a^u \alpha(s)\,ds\right)Y(a)-\int_a^b \alpha(t)\,Y(t)\,dt\right\| \tag{2.12}$$
$$\leq \sup_{t\in[a,b]}\left\|Y'(t)\right\|$$
$$\times\begin{cases} \left[(b-u)\int_u^b \left|\alpha(t)\right| dt+(u-a)\int_a^u \left|\alpha(t)\right| dt\right], \\ \\ \frac{1}{(q+1)^{1/q}}\left[(b-u)^{1+1/q}\left(\int_u^b \left|\alpha(t)\right|^p dt\right)^{1/p} \right. \\ \left. +(u-a)^{1+1/q}\left(\int_a^u \left|\alpha(t)\right|^p dt\right)^{1/p}\right], \\ \\ \frac{1}{2}\left[(b-u)^2\sup_{t\in[u,b]}\left|\alpha(t)\right|+(u-a)^2\sup_{t\in[a,u]}\left|\alpha(t)\right|\right] \end{cases}$$

for all $u\in[a,b]$.

Observe that

$$
\begin{aligned}
&(b-u)\int_u^b |\alpha(t)|\,dt + (u-a)\int_a^u |\alpha(t)|\,dt \\
&\le \max\{b-u, u-a\}\left[\int_u^b |\alpha(t)|\,dt + \int_a^u |\alpha(t)|\,dt\right] \\
&= \left[\frac{1}{2}(b-a) + \left|u-\frac{a+b}{2}\right|\right]\int_a^b |\alpha(t)|\,dt.
\end{aligned}
$$

By using the elementary inequality for $a,\ b,\ c,\ d \ge 0$ *and* $p,\ q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$,

$$
(ab+cd) \le (a^p + c^p)^{1/p}(b^q + d^q)^{1/q}
$$

we get

$$
\begin{aligned}
&(b-u)^{1+1/q}\left(\int_u^b |\alpha(t)|^p\,dt\right)^{1/p} + (u-a)^{1+1/q}\left(\int_a^u |\alpha(t)|^p\,dt\right)^{1/p} \\
&\le \left[\left((b-u)^{1+1/q}\right)^q + \left((u-a)^{1+1/q}\right)^q\right]^{1/q} \\
&\times \left[\left(\left(\int_u^b |\alpha(t)|^p\,dt\right)^{1/p}\right)^p + \left(\left(\int_a^u |\alpha(t)|^p\,dt\right)^{1/p}\right)^p\right]^{1/p} \\
&= \left[(b-u)^{q+1} + (u-a)^{q+1}\right]^{1/q}\left[\int_u^b |\alpha(t)|^p\,dt + \int_a^u |\alpha(t)|^p\,dt\right]^{1/p} \\
&= \left[(b-u)^{q+1} + (u-a)^{q+1}\right]^{1/q}\left(\int_a^b |\alpha(t)|^p\,dt\right)^{1/p}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\frac{1}{2}\left[(b-u)^2 \sup_{t\in[u,b]}|\alpha(t)| + (u-a)^2 \sup_{t\in[a,u]}|\alpha(t)|\right] \\
&\le \frac{1}{2}\left[(b-u)^2 + (u-a)^2\right]\sup_{t\in[a,b]}|\alpha(t)| \\
&= \left[\frac{1}{4}(b-a) + \left(u-\frac{a+b}{2}\right)^2\right]\sup_{t\in[a,b]}|\alpha(t)|.
\end{aligned}
$$

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Then by (2.12) *we get for* p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *that*

$$\left\| \left(\int_u^b \alpha(s)\, ds \right) Y(b) + \left(\int_a^u \alpha(s)\, ds \right) Y(a) - \int_a^b \alpha(t) Y(t)\, dt \right\| \tag{2.13}$$
$$\leq \sup_{t\in[a,b]} \left\| Y'(t) \right\|$$
$$\times \begin{cases} \left[\frac{1}{2}(b-a) + \left| u - \frac{a+b}{2} \right| \right] \int_a^b |\alpha(t)|\, dt, \\ \\ \frac{1}{(q+1)^{1/q}} \left[(b-u)^{q+1} + (u-a)^{q+1} \right]^{1/q} \left(\int_a^b |\alpha(t)|^p\, dt \right)^{1/p}, \\ \\ \left[\frac{1}{4}(b-a) + \left(u - \frac{a+b}{2} \right)^2 \right] \sup_{t\in[a,b]} |\alpha(t)| \end{cases}$$

for all $u \in [a, b]$.

We also have:

Corollary 2.6. *With the assumptions of Theorem* 2.1, *we have for all* $u \in [a, b]$,

$$\left\| \left(\int_u^b \alpha(s)\, ds \right) Y(b) + \left(\int_a^u \alpha(s)\, ds \right) Y(a) - \int_a^b \alpha(t) Y(t)\, dt \right\| \tag{2.14}$$
$$\leq \left[\left(\int_u^b |\alpha(s)|\, ds \right)^p (b-u) + \left(\int_a^u |\alpha(s)|\, ds \right)^p (u-a) \right]^{1/p}$$
$$\times \left(\int_a^b \left\| Y'(t) \right\|^q dt \right)^{1/q}$$
$$\leq (b-a)^{1/p} \left[\left(\int_u^b |\alpha(s)|\, ds \right)^p + \left(\int_a^u |\alpha(s)|\, ds \right)^p \right]^{1/p}$$
$$\times \left(\int_a^b \left\| Y'(t) \right\|^q dt \right)^{1/q}$$

for p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$

Proof. Observe that, by the elementary inequality for a, b, c, $d \geq 0$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$(ab + cd) \leq (a^p + c^p)^{1/p} (b^q + d^q)^{1/q},$$

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we have

$$\left[\int_u^b \left(\int_u^t |\alpha(s)|\,ds\right)^p dt\right]^{1/p} \left(\int_u^b \|Y'(t)\|^q\,dt\right)^{1/q}$$
$$+\left[\int_a^u \left(\int_t^u |\alpha(s)|\,ds\right)^p dt\right]^{1/p} \left(\int_a^u \|Y'(t)\|^q\,dt\right)^{1/q}$$
$$\le \left(\int_u^b \left(\int_u^t |\alpha(s)|\,ds\right)^p dt + \int_a^u \left(\int_t^u |\alpha(s)|\,ds\right)^p dt\right)^{1/p}$$
$$\times \left(\int_u^b \|Y'(t)\|^q\,dt + \int_a^u \|Y'(t)\|^q\,dt\right)^{1/q}$$
$$= \left(\int_u^b \left(\int_u^t |\alpha(s)|\,ds\right)^p dt + \int_a^u \left(\int_t^u |\alpha(s)|\,ds\right)^p dt\right)^{1/p}$$
$$\times \left(\int_a^b \|Y'(t)\|^q\,dt\right)^{1/q}$$

$$\le \left(\left(\int_u^b |\alpha(s)|\,ds\right)^p \int_u^b dt + \left(\int_a^u |\alpha(s)|\,ds\right)^p \int_a^u dt\right)^{1/p}$$
$$\times \left(\int_a^b \|Y'(t)\|^q\,dt\right)^{1/q}$$
$$= \left(\left(\int_u^b |\alpha(s)|\,ds\right)^p (b-u) + \left(\int_a^u |\alpha(s)|\,ds\right)^p (u-a)\right)^{1/p}$$
$$\times \left(\int_a^b \|Y'(t)\|^q\,dt\right)^{1/q}$$
$$\le (b-a)^{1/p} \left(\left(\int_u^b |\alpha(s)|\,ds\right)^p + \left(\int_a^u |\alpha(s)|\,ds\right)^p\right)^{1/p}$$
$$\times \left(\int_a^b \|Y'(t)\|^q\,dt\right)^{1/q},$$

which proves (2.14). □

Remark 2.7. *If $m \in (a,b)$ is such that (2.8) is valid, then by (2.14) we get*

$$\left\|\left(\int_m^b \alpha(s)\,ds\right) Y(b) + \left(\int_a^m \alpha(s)\,ds\right) Y(a) - \int_a^b \alpha(t)\,Y(t)\,dt\right\| \le \frac{1}{2}(b-a)^{1/p} \int_a^b |\alpha(s)|\,ds \left(\int_a^b \|Y'(t)\|^q\,dt\right)^{1/q}. \tag{2.15}$$

Assume that α, $Y : [a,b] \to E$, are continuous and Y is strongly differentiable on (a,b), then

$$\left\|\left(\int_{\frac{a+b}{2}}^b \alpha(s)\,ds\right) Y(b) + \left(\int_a^{\frac{a+b}{2}} \alpha(s)\,ds\right) Y(a) - \int_a^b \alpha(t)\,Y(t)\,dt\right\| \le C(\alpha, Y), \tag{2.16}$$

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where

$$C(\alpha,Y):=\int_{\frac{a+b}{2}}^{b}\left(\int_{\frac{a+b}{2}}^{t}|\alpha(s)|\,ds\right)\left\|Y'(t)\right\|dt+\int_{a}^{\frac{a+b}{2}}\left(\int_{t}^{\frac{a+b}{2}}|\alpha(s)|\,ds\right)\left\|Y'(t)\right\|dt.$$

We also have the bounds

$$\begin{aligned} C(\alpha,Y) &\\ \leq &\begin{cases} \int_{\frac{a+b}{2}}^{b}|\alpha(s)|\,ds\int_{\frac{a+b}{2}}^{b}\|Y'(t)\|\,dt, \\ \left[\int_{u}^{b}\left(\int_{\frac{a+b}{2}}^{t}|\alpha(s)|\,ds\right)^{p}dt\right]^{1/p}\left(\int_{\frac{a+b}{2}}^{b}\|Y'(t)\|^{q}\,dt\right)^{1/q}, \\ \int_{\frac{a+b}{2}}^{b}\left(\int_{\frac{a+b}{2}}^{t}|\alpha(s)|\,ds\right)dt\sup_{t\in\left[\frac{a+b}{2},b\right]}\|Y'(t)\|, \end{cases} \\ + &\begin{cases} \left(\int_{a}^{\frac{a+b}{2}}|\alpha(s)|\,ds\right)\int_{a}^{\frac{a+b}{2}}\|Y'(t)\|\,dt, \\ \left[\int_{a}^{\frac{a+b}{2}}\left(\int_{t}^{\frac{a+b}{2}}|\alpha(s)|\,ds\right)^{p}dt\right]^{1/p}\left(\int_{a}^{\frac{a+b}{2}}\|Y'(t)\|^{q}\,dt\right)^{1/q}, \\ \int_{a}^{\frac{a+b}{2}}\left(\int_{t}^{\frac{a+b}{2}}|\alpha(s)|\,ds\right)dt\sup_{t\in\left[a,\frac{a+b}{2}\right]}\|Y'(t)\|, \end{cases} \end{aligned} \tag{2.17}$$

where p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

From (2.7) we get

$$\begin{aligned} &\left\|\left(\int_{\frac{a+b}{2}}^{b}\alpha(s)\,ds\right)Y(b)+\left(\int_{a}^{\frac{a+b}{2}}\alpha(s)\,ds\right)Y(a)-\int_{a}^{b}\alpha(t)\,Y(t)\,dt\right\| \\ &\leq\int_{\frac{a+b}{2}}^{b}|\alpha(s)|\,ds\int_{\frac{a+b}{2}}^{b}\left\|Y'(t)\right\|dt+\int_{a}^{\frac{a+b}{2}}|\alpha(s)|\,ds\int_{a}^{\frac{a+b}{2}}\left\|Y'(t)\right\|dt \\ &\leq\begin{cases} \max\left\{\int_{\frac{a+b}{2}}^{b}|\alpha(s)|\,ds,\int_{a}^{\frac{a+b}{2}}|\alpha(s)|\,ds\right\}\int_{a}^{b}\|Y'(t)\|\,dt, \\ \int_{a}^{b}|\alpha(s)|\,ds\max\left\{\int_{\frac{a+b}{2}}^{b}\|Y'(t)\|\,dt,\int_{a}^{\frac{a+b}{2}}\|Y'(t)\|\,dt\right\}, \end{cases} \\ &\leq\int_{a}^{b}|\alpha(s)|\,ds\int_{a}^{b}\left\|Y'(t)\right\|dt, \end{aligned} \tag{2.18}$$

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while from (2.13) we get

$$\left\| \left(\int_{\frac{a+b}{2}}^{b} \alpha(s)\, ds \right) Y(b) + \left(\int_{a}^{\frac{a+b}{2}} \alpha(s)\, ds \right) Y(a) - \int_a^b \alpha(t)\, Y(t)\, dt \right\| \tag{2.19}$$

$$\leq \sup_{t\in[a,b]} \left\| Y'(t) \right\| \times \begin{cases} \frac{1}{2}(b-a) \int_a^b |\alpha(s)|\, dt, \\ \\ \frac{1}{2(q+1)^{1/q}} (b-a)^{1+1/q} \left(\int_a^b |\alpha(s)|^p\, dt \right)^{1/p}, \\ \\ \frac{1}{4}(b-a) \sup_{t\in[a,b]} |\alpha(s)|. \end{cases}$$

From (2.14) we also get

$$\begin{aligned} &\left\| \left(\int_{\frac{a+b}{2}}^{b} \alpha(s)\, ds \right) Y(b) + \left(\int_{a}^{\frac{a+b}{2}} \alpha(s)\, ds \right) Y(a) - \int_a^b \alpha(t)\, Y(t)\, dt \right\| \\ &\leq \frac{(b-a)^{1/p}}{2^{1/p}} \left[\left(\int_{\frac{a+b}{2}}^{b} |\alpha(s)|\, ds \right)^p + \left(\int_a^{\frac{a+b}{2}} |\alpha(s)|\, ds \right)^p \right]^{1/p} \\ &\times \left(\int_a^b \left\| Y'(t) \right\|^q dt \right)^{1/q}. \end{aligned} \tag{2.20}$$

If we consider the case when $\alpha(t) = 1$, $t \in [a,b]$, then by (2.1) we get for all $u \in [a,b]$,

$$\left\| (b-u)\, Y(b) + (u-a)\, Y(a) - \int_a^b Y(t)\, dt \right\| \leq C(Y,u), \tag{2.21}$$

where

$$C(Y,u) := \int_u^b (t-u) \left\| Y'(t) \right\| dt + \int_a^u (u-t) \left\| Y'(t) \right\| dt.$$

We also have the bounds

$$\begin{aligned} C(Y,u) \leq &\begin{cases} (b-u) \int_u^b \|Y'(t)\|\, dt, \\ \\ \frac{(b-u)^{1+1/p}}{(p+1)^{1/p}} \left(\int_u^b \|Y'(t)\|^q\, dt \right)^{1/q}, \\ \\ \frac{1}{2}(b-u)^2 \sup_{t\in[u,b]} \|Y'(t)\|, \end{cases} \\ + &\begin{cases} (u-a) \int_a^u \|Y'(t)\|\, dt, \\ \\ \frac{(u-a)^{1+1/p}}{(p+1)^{1/p}} \left(\int_a^u \|Y'(t)\|^q\, dt \right)^{1/q}, \\ \\ \frac{1}{2}(u-a)^2 \sup_{t\in[a,u]} \|Y'(t)\|, \end{cases} \end{aligned} \tag{2.22}$$

for all $u \in [a,b]$, where p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

From (2.7) we get

$$\begin{aligned}
&\left\|(b-u)\,Y\,(b)+(u-a)\,Y\,(a)-\int_a^b Y\,(t)\,dt\right\| \\
&\le (b-u)\int_u^b \left\|Y'\,(t)\right\|dt+(u-a)\int_a^u \left\|Y'\,(t)\right\|dt \\
&\le \left[\frac{1}{2}\,(b-a)+\left|u-\frac{a+b}{2}\right|\right]\int_a^b \left\|Y'\,(t)\right\|dt
\end{aligned} \tag{2.23}$$

for all $u\in[a,b]$.

From (2.14) we get

$$\begin{aligned}
&\left\|(b-u)\,Y\,(b)+(u-a)\,Y\,(a)-\int_a^b Y\,(t)\,dt\right\| \\
&\le \left[(b-u)^{p+1}+(u-a)^{p+1}\right]^{1/p}\left(\int_a^b \left\|Y'\,(t)\right\|^q dt\right)^{1/q}
\end{aligned} \tag{2.24}$$

for all $u\in[a,b]$.

We also have the dual result:

Theorem 2.8. *Assume that $\alpha:[a,b]\to\mathbb{C}$, $Y:[a,b]\to E$ are continuous and α is continuously differentiable on (a,b), then for all $u\in[a,b]$*

$$\begin{aligned}
&\left\|\alpha\,(b)\left(\int_u^b Y\,(s)\,ds\right)+\alpha\,(a)\left(\int_a^u Y\,(s)\,ds\right)-\int_a^b \alpha\,(t)\,Y\,(t)\,dt\right\| \\
&\le \widetilde{C}\,(\alpha,Y,u)\,,
\end{aligned} \tag{2.25}$$

where

$$\widetilde{C}\,(\alpha,Y,u):=\int_u^b\left(\int_u^t \left\|Y\,(s)\right\|ds\right)\left|\alpha'\,(t)\right|dt+\int_a^u\left(\int_t^u \left\|Y\,(s)\right\|ds\right)\left|\alpha'\,(t)\right|dt.$$

We also have the bounds

$$\begin{aligned}
&\widetilde{C}\,(\alpha,Y,u) \\
&\le \begin{cases}
\int_u^b \left\|Y\,(s)\right\|ds\int_u^b \left|\alpha'\,(t)\right|dt, \\
\\
\left[\int_u^b\left(\int_u^t \left\|Y\,(s)\right\|ds\right)^p dt\right]^{1/p}\left(\int_u^b \left|\alpha'\,(t)\right|^q dt\right)^{1/q}, \\
\\
\int_u^b\left(\int_u^t \left\|Y\,(s)\right\|ds\right)dt\,\sup_{t\in[u,b]}\left|\alpha'\,(t)\right|,
\end{cases} \\
&+\begin{cases}
\left(\int_a^u \left\|Y\,(s)\right\|ds\right)\int_a^u \left|\alpha'\,(t)\right|dt, \\
\\
\left[\int_a^u\left(\int_t^u \left\|Y\,(s)\right\|ds\right)^p dt\right]^{1/p}\left(\int_a^u \left|\alpha'\,(t)\right|^q dt\right)^{1/q}, \\
\\
\int_a^u\left(\int_t^u \left\|Y\,(s)\right\|ds\right)dt\,\sup_{t\in[a,u]}\left|\alpha'\,(t)\right|,
\end{cases}
\end{aligned} \tag{2.26}$$

where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

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Proof. Observe that the following identity of interest holds

$$\begin{aligned}\alpha(b)\left(\int_u^b Y(s)\,ds\right)+\alpha(a)\left(\int_a^u Y(s)\,ds\right)-\int_a^b \alpha(t)\,Y(t)\,dt \\ =\int_u^b \alpha'(t)\left(\int_u^t Y(s)\,ds\right)dt-\int_a^u \alpha'(t)\left(\int_t^u Y(s)\,ds\right)dt\end{aligned} \tag{2.27}$$

for all $u\in[a,b]$.

Indeed, using integration by parts in Bochner's integral, we get

$$\begin{aligned}\int_u^b \alpha'(t)\left(\int_u^t Y(s)\,ds\right)dt &= \alpha(t)\left(\int_u^t Y(s)\,ds\right)\Big|_u^b-\int_u^b \alpha(t)\,Y(t)\,dt \\ &=\alpha(b)\int_u^b Y(s)\,ds-\int_u^b \alpha(t)\,Y(t)\,dt\end{aligned}$$

and

$$\begin{aligned}\int_a^u \alpha'(t)\left(\int_t^u Y(s)\,ds\right)dt &= \alpha(t)\left(\int_t^u Y(s)\,ds\right)\Big|_a^u+\int_a^u \alpha(t)\,Y(t)\,dt \\ &=-\alpha(a)\left(\int_a^u Y(s)\,ds\right)+\int_a^u \alpha(t)\,Y(t)\,dt.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_u^b \alpha'(t)\left(\int_u^t Y(s)\,ds\right)dt-\int_a^u \alpha'(t)\left(\int_t^u Y(s)\,ds\right) \\ &=\alpha(b)\int_u^b Y(s)\,ds-\int_u^b \alpha(t)\,Y(t)\,dt+\alpha(a)\left(\int_a^u Y(s)\,ds\right)-\int_a^u \alpha(t)\,Y(t)\,dt \\ &=\alpha(b)\int_u^b Y(s)\,ds+\alpha(a)\left(\int_a^u Y(s)\,ds\right)-\int_a^b \alpha(t)\,Y(t)\,dt\end{aligned}$$

and the identity (2.27).

The rest follows in a similar way to the one in the proof of Theorem 2.1 and the details are omitted. □

Similar consequence may be obtained for the dual case as well. However we do not state them here.

3. Inequalities for operator monotone functions

A real valued continuous function h on $[0,\infty)$ is said to be operator monotone if $h(A)\geq h(B)$ holds for any $A\geq B\geq 0$ operators on the Hilbert space H.

We have the following representation of operator monotone functions, see for instance [2, p. 144-145]:

Theorem 3.1. *A function $h:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$h(t)=h(0)+bt+\int_0^\infty \frac{t\lambda}{t+\lambda}d\mu(\lambda), \tag{3.1}$$

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where $b \geq 0$ and a positive measure μ on $[0, \infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \tag{3.2}$$

We have the following representation result:

Lemma 3.2. *Let $h : [0, \infty) \to \mathbb{R}$ be operator monotone in $[0, \infty)$. Assume that $U \geq 0$, then for all selfadjoint operators V we have*

$$Dh(U)(V) = bV + \int_0^\infty \lambda^2 \left[(\lambda + U)^{-1} V (\lambda + U)^{-1} \right] d\mu(\lambda). \tag{3.3}$$

Proof. From (3.1) we get

$$h(t) = h(0) + bt + \int_0^\infty \left(\lambda - \frac{\lambda^2}{t+\lambda} \right) d\mu(\lambda).$$

Assume that $U \geq 0$, then for all selfadjoint operator V we have, by the representation of h and for t in a small open interval around 0, that

$$\begin{aligned}
&h(U+tV) - h(U) \\
&= btV + \int_0^\infty \left(\lambda - \lambda^2 (U + tV + \lambda)^{-1} \right) d\mu(\lambda) - \int_0^\infty \left(\lambda - \lambda^2 (U+\lambda)^{-1} \right) d\mu(\lambda) \\
&= btV + \int_0^\infty \lambda^2 \left[(\lambda + U)^{-1} - (\lambda + U + tV)^{-1} \right] d\mu(\lambda) \\
&= btV + \int_0^\infty \lambda^2 \left[(\lambda + U)^{-1} (\lambda + U + tV - \lambda - U) (\lambda + U + tV)^{-1} \right] d\mu(\lambda) \\
&= btV + t \int_0^\infty \lambda^2 \left[(\lambda + U)^{-1} V (\lambda + U + tV)^{-1} \right] d\mu(\lambda).
\end{aligned}$$

Dividing by $t \neq 0$, we get

$$\frac{h(U+tV) - h(U)}{t} = bV + \int_0^\infty \lambda^2 \left[(\lambda + U)^{-1} V (\lambda + U + tV)^{-1} \right] d\mu(\lambda)$$

and by taking the limit over $t \to 0$, we get

$$Dh(U)(V) = bV + \int_0^\infty \lambda^2 \left[(\lambda + U)^{-1} V (\lambda + U)^{-1} \right] d\mu(\lambda)$$

for all selfadjoint operator V we have (3.3). □

Lemma 3.3. *Let $h : [0, \infty) \to \mathbb{R}$ be operator monotone in $[0, \infty)$. Assume that $U \geq u > 0$, then for all selfadjoint operators V we have*

$$\| Dh(U)(V) \| \leq h'(u) \| V \|. \tag{3.4}$$

Proof. From (3.3) we get

$$\begin{aligned}
\| Dh(U)(V) - bV \| &\leq \int_0^\infty \lambda^2 \left\| (\lambda + U)^{-1} V (\lambda + U)^{-1} \right\| d\mu(\lambda) \\
&\leq \| V \| \int_0^\infty \lambda^2 \left\| (\lambda + U)^{-1} \right\|^2 d\mu(\lambda).
\end{aligned} \tag{3.5}$$

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Observe that $\lambda + U \geq \lambda + u > 0$ for $\lambda \in [0, \infty)$. Then $0 < (\lambda + U)^{-1} \leq (\lambda + u)^{-1}$, which implies that $\left\| (\lambda + U)^{-1} \right\| \leq (\lambda + u)^{-1}$, namely $\left\| (\lambda + U)^{-1} \right\|^2 \leq (\lambda + u)^{-2}$ for $\lambda \in [0, \infty)$.

Therefore by (3.5) we get

$$\| Dh(U)(V) - bV \| \leq \| V \| \int_0^\infty \lambda^2 (\lambda + u)^{-2} \, d\mu(\lambda) . \tag{3.6}$$

If we take the derivative over t in (3.1) then we have

$$h'(t) = b + \int_0^\infty \frac{\lambda(t+\lambda) - \lambda t}{(t+\lambda)^2} d\mu(\lambda) = b + \int_0^\infty \frac{\lambda^2}{(t+\lambda)^2} d\mu(\lambda) \tag{3.7}$$

for $t > 0$.

From (3.7) we get

$$\int_0^\infty \lambda^2 (\lambda + u)^{-2} \, d\mu(\lambda) = h'(u) - b,$$

and by (3.6) we derive

$$\| Dh(U)(V) - bV \| \leq \| V \| \, h'(u) - b \, \| V \| .$$

Finally, by the triangle inequality and by the fact that $b \geq 0$, we obtain that

$$\| Dh(U)(V) \| - b \, \| V \| \leq \| Dh(U)(V) - bV \| ,$$

which proves the desired result (3.4). $\square$

For a continuous real valued function h on $(0, \infty)$ and two given operators A, $B > 0$ we consider the auxiliary function $h_{A,B}$ defined by the use of continuous funcional calculus for selfadjoint operators,

$$h_{A,B}(t) := h((1-t)A + tB), \ t \in [0,1] .$$

We have the following representations of the derivatives:

Lemma 3.4. *Assume that the operator function generated by h is Fréchet differentiable in each $A \geq 0$, then for $B \geq 0$ we have that $h_{A,B}$ is differentiable on $[0,1]$ and*

$$h'_{A,B}(t) = D(h)((1-t)A + tB)(B - A) \tag{3.8}$$

for $t \in [0,1]$, where in 0 and 1 the derivatives are the right and left derivatives.

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t + h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned} & \frac{h_{A,B}(t+h) - h(t)}{h} \\ & = \frac{h((1-(t+h))A + (t+h)B) - h((1-t)A + tB)}{h} \\ & = \frac{h((1-t)A + tB + h(B-A)) - h((1-t)A + tB)}{h} \end{aligned}$$

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and by taking the limit over $h \to 0$, we get

$$\begin{aligned} h'_{A,B}(t) &= \lim_{h\to 0} \frac{h_{A,B}(t+h) - h(t)}{h} \\ &= \lim_{h\to 0} \left[\frac{h((1-t)A + tB + h(B-A)) - h((1-t)A + tB)}{h} \right] \\ &= D(h)((1-t)A + tB)(B-A), \end{aligned}$$

which proves (3.8). □

Corollary 3.5. *Let $h : [0, \infty) \to \mathbb{R}$ be operator monotone in $[0, \infty)$. Then for all $A \geq a > 0$, $B \geq b > 0$ we have*

$$\begin{aligned} \left\| h'_{A,B}(t) \right\| &= \| D(h)((1-t)A + tB)(B-A) \| \\ &\leq h'((1-t)a + tb) \| B - A \| \end{aligned} \tag{3.9}$$

for all $t \in [0, 1]$.

The proof follows by Lemma 3.3 and Lemma 3.4.

One can observe that the inequality (3.9) remains valid for operator monotone functions on $(0, \infty)$. This follows by considering the function $h_\varepsilon(t) := h(t + \varepsilon)$ for $\varepsilon > 0$, which is operator monotone on $[0, \infty)$ and then by letting $\varepsilon \to 0+$ and using the continuity of h and h'.

Theorem 3.6. *Let $f : [0, \infty) \to \mathbb{R}$ be operator monotone in $[0, \infty)$ and $\alpha : [0, 1] \to \mathbb{C}$ a continuous function on $[0, 1]$. Then for all $A \geq a > 0$, $B \geq b > 0$ we have*

$$\begin{aligned} &\left\| \left(\int_u^1 \alpha(s)\, ds \right) f(A) + \left(\int_0^u \alpha(s)\, ds \right) f(B) \right. \\ &\left. - \int_0^1 \alpha(t) f((1-t)A + tB)\, dt \right\| \\ &\leq \| B - A \| \left[\int_u^1 |\alpha(s)|\, ds \int_u^1 f'((1-t)a + tb)\, dt \right. \\ &\left. + \int_0^u |\alpha(s)|\, ds \int_0^u f'((1-t)a + tb)\, dt \right] \\ &\leq \| B - A \| \max\left\{ \int_u^b |\alpha(s)|\, ds, \int_0^u |\alpha(s)|\, ds \right\} \\ &\times \begin{cases} \frac{f(b) - f(a)}{b - a} & \text{if } b \neq a, \\ \\ f'(a) & \text{if } b = a, \end{cases} \\ &\leq \| B - A \| \int_0^1 |\alpha(s)|\, ds \times \begin{cases} \frac{f(b) - f(a)}{b - a} & \text{if } b \neq a, \\ \\ f'(a) & \text{if } b = a, \end{cases} \end{aligned} \tag{3.10}$$

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$$\begin{aligned}&\left\|\left(\int_u^1 \alpha(s)\,ds\right) f(A)+\left(\int_0^u \alpha(s)\,ds\right) f(B)\right.\\&\left.-\int_0^1 \alpha(t)\, f((1-t)A+tB)\,dt\right\|\\&\leq \|B-A\|\left[\int_u^1 (1-t)\,|\alpha(t)|\,dt+\int_0^u t\,|\alpha(t)|\,dt\right]\sup_{t\in[0,1]} f'((1-t)a+tb)\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}&\left\|\left(\int_u^1 \alpha(s)\,ds\right) f(A)+\left(\int_0^u \alpha(s)\,ds\right) f(B)\right.\\&\left.-\int_0^1 \alpha(t)\, f((1-t)A+tB)\,dt\right\|\\&\leq \|B-A\|\left[\left(\int_u^1 |\alpha(s)|\,ds\right)^p(1-u)+\left(\int_0^u |\alpha(s)|\,ds\right)^p u\right]^{1/p}\\&\times\left(\int_0^1 \left[f'((1-t)a+tb)\right]^q dt\right)^{1/q}\\&\leq \|B-A\|\left[\left(\int_u^1 |\alpha(s)|\,ds\right)^p+\left(\int_0^u |\alpha(s)|\,ds\right)^p\right]^{1/p}\\&\times\left(\int_0^1 \left[f'((1-t)a+tb)\right]^q dt\right)^{1/q}\end{aligned} \tag{3.12}$$

for $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$

Proof. We use inequality (2.7) for α and $f_{A,B}$ on $[0,1]$ to get

$$\begin{aligned}&\left\|\left(\int_u^1 \alpha(s)\,ds\right) f_{A,B}(1)+\left(\int_0^u \alpha(s)\,ds\right) f_{A,B}(0)-\int_0^1 \alpha(t)\, f_{A,B}(t)\,dt\right\|\\&\leq \int_u^1 |\alpha(s)|\,ds\int_u^1 \left\|f'_{A,B}(t)\right\|dt+\int_0^u |\alpha(s)|\,ds\int_0^u \left\|f'_{A,B}(t)\right\|dt\\&\leq \max\left\{\int_u^b |\alpha(s)|\,ds,\int_0^u |\alpha(s)|\,ds\right\}\int_0^1 \left\|f'_{A,B}(t)\right\|dt\\&\leq \int_0^1 |\alpha(s)|\,ds\int_0^1 \left\|f'_{A,B}(t)\right\|dt,\end{aligned} \tag{3.13}$$

and, since by (3.9)

$$\left\|f'_{A,B}(t)\right\|\leq f'((1-t)a+tb)\,\|B-A\|$$

for all $t\in[0,1]$, hence by (3.13) we get (3.10).

The inequality (3.11) follows in a similar way from (2.10) while (3.12) follows by (2.14). □

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If we take in Theorem 3.6 $f(t)=t^r$, $r\in(0,1)$, which is operator monotone on $[0,\infty)$, then for all $A\geq a>0$, $B\geq b>0$ we have for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ that

$$\begin{aligned}(3.14)\qquad &\left\|\left(\int_u^1\alpha(s)\,ds\right)A^r+\left(\int_0^u\alpha(s)\,ds\right)B^r\right.\\&\left.-\int_0^1\alpha(t)\left((1-t)A+tB\right)^r dt\right\|\\&\leq\|B-A\|\max\left\{\int_u^b|\alpha(s)|\,ds,\int_0^u|\alpha(s)|\,ds\right\}\\&\times\begin{cases}\frac{b^r-a^r}{b-a}\text{ if } b\neq a,\\ \\ ra^{r-1}\text{ if } b=a,\end{cases}\\&\leq\|B-A\|\int_0^1|\alpha(s)|\,ds\times\begin{cases}\frac{b^r-a^r}{b-a}\text{ if } b\neq a,\\ \\ ra^{r-1}\text{ if } b=a,\end{cases}\end{aligned}$$

$$\begin{aligned}(3.15)\qquad &\left\|\left(\int_u^1\alpha(s)\,ds\right)A^r+\left(\int_0^u\alpha(s)\,ds\right)B^r\right.\\&\left.-\int_0^1\alpha(t)\left((1-t)A+tB\right)^r dt\right\|\\&\leq r\max\left\{a^{r-1},b^{r-1}\right\}\|B-A\|\left[\int_u^1(1-t)|\alpha(t)|\,dt+\int_0^u t|\alpha(t)|\,dt\right]\end{aligned}$$

and

$$\begin{aligned}(3.16)\qquad &\left\|\left(\int_u^1\alpha(s)\,ds\right)A^r+\left(\int_0^u\alpha(s)\,ds\right)B^r\right.\\&\left.-\int_0^1\alpha(t)\left((1-t)A+tB\right)^r dt\right\|\\&\leq r\|B-A\|\left[\left(\int_u^1|\alpha(s)|\,ds\right)^p(1-u)+\left(\int_0^u|\alpha(s)|\,ds\right)^p u\right]^{1/p}\\&\times\left(\int_0^1\left((1-t)a+tb\right)^{q(r-1)}dt\right)^{1/q}\\&\leq r\|B-A\|\left[\left(\int_u^1|\alpha(s)|\,ds\right)^p+\left(\int_0^u|\alpha(s)|\,ds\right)^p\right]^{1/p}\\&\times\left(\int_0^1\left((1-t)a+tb\right)^{q(r-1)}dt\right)^{1/q}.\end{aligned}$$

If we consider the operator monotone function $f(t) = \ln t$, $t \in (0, \infty)$ in Theorem 3.6, then we get

$$\begin{aligned}
(3.17)\quad & \left\| \left(\int_u^1 \alpha(s)\,ds \right) \ln A + \left(\int_0^u \alpha(s)\,ds \right) \ln B \right. \\
& \left. - \int_0^1 \alpha(t) \ln((1-t)A + tB)\,dt \right\| \\
& \le \|B - A\| \max \left\{ \int_u^b |\alpha(s)|\,ds, \int_0^u |\alpha(s)|\,ds \right\} \\
& \times \begin{cases} \frac{\ln b - \ln a}{b-a} \text{ if } b \ne a, \\ \\ a^{-1} \text{ if } b = a, \end{cases} \\
& \le \|B - A\| \int_0^1 |\alpha(s)|\,ds \times \begin{cases} \frac{\ln b - \ln a}{b-a} \text{ if } b \ne a, \\ \\ a^{-1} \text{ if } b = a, \end{cases}
\end{aligned}$$

$$\begin{aligned}
(3.18)\quad & \left\| \left(\int_u^1 \alpha(s)\,ds \right) \ln A + \left(\int_0^u \alpha(s)\,ds \right) \ln B \right. \\
& \left. - \int_0^1 \alpha(t) \ln((1-t)A + tB)\,dt \right\| \\
& \le \frac{1}{\min\{a,b\}} \|B - A\| \left[\int_u^1 (1-t)|\alpha(t)|\,dt + \int_0^u t|\alpha(t)|\,dt \right]
\end{aligned}$$

and

$$\begin{aligned}
(3.19)\quad & \left\| \left(\int_u^1 \alpha(s)\,ds \right) \ln A + \left(\int_0^u \alpha(s)\,ds \right) \ln B \right. \\
& \left. - \int_0^1 \alpha(t) \ln((1-t)A + tB)\,dt \right\| \\
& \le \|B - A\| \left[(1-u) \left(\int_u^1 |\alpha(s)|\,ds \right)^p + u \left(\int_0^u |\alpha(s)|\,ds \right)^p \right]^{1/p} \\
& \times \begin{cases} \left(\frac{b^{1-q} - a^{1-q}}{(1-q)(b-a)} \right)^{1/q} \text{ if } b \ne a, \\ \\ a^{-1} \text{ if } b = a, \end{cases} \\
& \le \|B - A\| \left[\left(\int_u^1 |\alpha(s)|\,ds \right)^p + \left(\int_0^u |\alpha(s)|\,ds \right)^p \right]^{1/p} \\
& \times \begin{cases} \left(\frac{b^{1-q} - a^{1-q}}{(1-q)(b-a)} \right)^{1/q} \text{ if } b \ne a, \\ \\ a^{-1} \text{ if } b = a \end{cases}
\end{aligned}$$

for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

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