APPROXIMATE BIPROJECTIVITY AND BIFLATNESS OF SOME ALGEBRAS OVER CERTAIN SEMIGROUPS

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ABSTRACT. We investigate (bounded) approximate biprojectivity of $l^1(S)$ for uniformly locally finite inverse semigroups. As a consequence, we show that when $S = \mathcal{M}(G, I)$ is the Brandt inverse semigroup, then $l^1(S)$ is (boundedly) approximately biprojective if and only if G is amenable. Moreover, we study biflatness and (bounded) approximate biprojectivity of the measure algebra $M(S)$ of a topological Brandt semigroup.

1. Introduction

In [8], Ramsden has characterized biprojectivity and biflatness of the semigroup algebra $l^1(S)$ for an inverse semigroup S. Motivated by this investigation, in this paper, we verify (bounded) approximate biprojectivity of $l^1(S)$ when S is a uniformly locally finite inverse semigroup. Then we characterize (bounded) approximate biprojectivity of $l^1(S)$ for the Brandt inverse semigroup $S = \mathcal{M}(G, I)$. We also characterize (bounded) approximate biprojectivity of the Fourier algebra $A(S)$ of a Clifford semigroup S. In the last section, we study biflatness and approximate biprojectivity of some Banach algebras related to topological Brandt semigroups. First we recall briefly definitions.

A Banach space X is said to have the (*bounded*) *approximation property* if there exists a (bounded) net (T_α) of finite rank operators on X such that $T_\alpha \to I_X$ uniformly on compact subsets of X, where I_X is the identity map on X.

For two Banach spaces X and Y, we denote by $X \hat{\otimes} Y$ their projective tensor product and by $\mathcal{B}(X, Y)$ the space of all bounded linear operators from X to Y. Suppose X_1, X_2, Y_1 and Y_2 are

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Banach spaces and $T_i \in \mathcal{B}(X_i, Y_i)$ for $i = 1, 2$. The tensor product of T_1 and T_2 is the linear map defined by $T_1 \otimes T_2 : X_1 \hat{\otimes} X_2 \to Y_1 \hat{\otimes} Y_2$ by

$$(T_1 \otimes T_2)(x_1 \otimes x_2) = T_1(x_1) \otimes T_2(x_2) \qquad (x_1 \in X_1, x_2 \in X_2).$$

The projective tensor product of two Banach algebras A and B is denoted by $A \hat{\otimes} B$, which is a Banach algebra with the following multiplication:

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2) \qquad (a_1, a_2 \in A, b_1, b_2 \in B).$$

Let A be a Banach algebra. The product map on A extends to a map $\Delta_A : A \hat{\otimes} A \to A$, determined by $\Delta_A(a \otimes b) = ab$ for all $a, b \in A$. The projective tensor product $A \hat{\otimes} A$ becomes a Banach A-bimodule with the following module actions:

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \qquad (a, b, c \in A).$$

By these actions Δ_A becomes an A-bimodule homomorphism.

An *approximate diagonal* for A is a net $(\mathbf{m}_\alpha) \subseteq A \hat{\otimes} A$ such that $a \cdot \mathbf{m}_\alpha - \mathbf{m}_\alpha \cdot a \to 0$ and $(\Delta_A \mathbf{m}_\alpha)a \to a$ for each $a \in A$. A Banach algebra A is called *pseudo-amenable* if it has an approximate diagonal ([5]).

A Banach algebra A is called *biprojective* if Δ_A has a bounded right inverse which is an A-bimodule homomorphism and is called *biflat* if Δ_A^* has a bounded left inverse which is an A-bimodule homomorphism. Different notations of approximate biprojectivity are considered in [1,7] and [12]. We use the given one by the first author in [7] that includes much more classes of Banach algebras; see [7].

Definition 1.1. A Banach algebra A is called *approximately biprojective* if there exists a net $(\rho_\alpha) \subset \mathcal{B}(A, A \hat{\otimes} A)$ such that

   (i) $\Delta_A \circ \rho_\alpha(a) \to a$ for all $a \in A$,
   (ii) $a \cdot \rho_\alpha(b) - \rho_\alpha(ab) \to 0$ for all $a, b \in A$,
   (iii) $\rho_\alpha(a) \cdot b - \rho_\alpha(ab) \to 0$ for all $a, b \in A$.

Moreover, if the net (ρ_α) can be chosen to be bounded with bound C, we say that A is *boundedly approximately biprojective* or C-*approximately biprojective*. The infimum of such $C > 0$ is called the *approximate biprojectivity constant* of A and is denoted by $\mathrm{BC}(A)$.

Notice that Zhang assumes each ρ_α in Definition 1.1 is an A-bimodule homomorphism and Aristov assumes the convergence in conditions (i), (ii) and (iii) is the uniformly convergence on compact subsets of A; [1, Proposition 5.5]. We remark that our definition coincides with Aristov's definition in bounded case; see [1, Theorem 7.2]. Note also that if A has the bounded approximation property, by [1, Theorems 3.6(B) and 7.3(C)], bounded approximate biprojectivity is equivalent to biflatness.

A (discrete) semigroup S is called an *inverse semigroup* if for each $s \in S$ there is a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. An element $e \in S$ is called an *idempotent* if $e^2 = e^* = e$. The set of all idempotents of S is denoted by $E(S)$. For each $e \in E(S)$ we define $G_e = \{s \in S : ss^* = s^*s = e\}$, which is a maximal subgroup of S. An inverse semigroup S is called a *Clifford semigroup* if $s^*s = ss^*$ for all $s \in S$.

2. **Approximate biprojectivity of some semigroup algebras**

In the following proposition we characterize (bounded) approximate biprojectivity of l^1-direct sum of Banach algebras. Let $(A_i)_{i \in I}$ be a family of Banach algebras. Then the l^1-direct sum of $(A_i)_{i \in I}$ is defined by

$$A = l^1 - \bigoplus_{i \in I} A_i = \left\{ a = (a_i)_{i \in I} : \|a\| = \sum_{i \in I} \|a_i\|_{A_i} < \infty \right\}.$$

Obviously, A is a Banach algebra with componentwise product.

Proposition 2.1. *Let $(A_i)_{i \in I}$ be a family of Banach algebras and let $A = l^1 - \bigoplus_{i \in I} A_i$. Then*

- *(i) A is approximately biprojective if and only if each A_i is approximately biprojective.*
- *(ii) A is C-approximately biprojective if and only if each A_i is C-approximately biprojective.*

Proof. Let A be approximately biprojective and $(\rho_\lambda) \subset \mathcal{B}(A, A \hat{\otimes} A)$ be a net satisfying Definition 1.1. Let also for each $i \in I$, $\iota_i : A_i \to A$ and $p_i : A \to A_i$ be the canonical injection and projection, respectively. Set $\rho_{i,\lambda} = (p_i \otimes p_i) \circ \rho_\lambda \circ \iota_i \in \mathcal{B}(A_i, A_i \hat{\otimes} A_i)$. Then for each $a_i \in A_i$,

$$\lim_\lambda \Delta_{A_i} \circ \rho_{i,\lambda}(a_i) = \lim_\lambda \Delta_{A_i} \circ (p_i \otimes p_i) \circ \rho_\lambda \circ \iota_i(a_i) = \lim_\lambda p_i \circ \Delta_A \circ \rho_\lambda(\iota_i(a_i)) = p_i(\iota_i(a_i)) = a_i,$$

and

$$\lim_\lambda \rho_{i,\lambda}(a_i b_i) - a_i \rho_{i,\lambda}(b_i) = \lim_\lambda (p_i \otimes p_i)[\rho_\lambda(\iota_i(a_i)\iota_i(b_i)) - \iota_i(a_i)\rho_\lambda(\iota_i(b_i))] = 0,$$

and likewise, $\lim_\lambda \rho_{i,\lambda}(a_i b_i) - \rho_{i,\lambda}(a_i) b_i = 0$. So each A_i is approximately biprojective.

Conversely, assume that each A_i is approximately biprojective. Take $\epsilon > 0$, $F = \{a_1, \ldots, a_n\} \subset A$ and $G = \{b_1, \ldots, b_n\} \subset A$. Suppose $a_k = (a_{k,i})$ and $b_k = (b_{k,i})$ for $1 \le k \le n$. Then there is a finite subset J of I such that

$$\sum_{i \notin J} \|a_{k,i}\| < \epsilon \quad \text{and} \quad \sum_{i \notin J} \|b_{k,i}\| < \epsilon \qquad (1 \le k \le n).$$

Now for all $i \in J$ there exist $\lambda = \lambda(F, G, \epsilon)$ and $\rho_{i,\lambda} : A_i \to A_i \hat{\otimes} A_i$ such that

$$\|\Delta_{A_i} \circ \rho_{i,\lambda}(a_{k,i}) - a_{k,i}\| < \frac{\epsilon}{\# J},$$

and

$$\|\rho_{i,\lambda}(a_{k,i} b_{k,i}) - a_{k,i}\rho_{i,\lambda}(b_{k,i})\| < \frac{\epsilon}{\# J}, \quad \|\rho_{i,\lambda}(a_{k,i} b_{k,i}) - \rho_{i,\lambda}(a_{k,i}) b_{k,i}\| < \frac{\epsilon}{\# J}.$$

Define $\rho_\lambda = \rho_{\lambda(F,G,\epsilon)} := l^1 - \oplus_{i \in J} \rho_{i,\lambda(F,G,\epsilon)}$. Then for each $1 \le k \le n$,

$$\|\Delta_A \circ \rho_\lambda(a_k) - a_k\| \le \|\Delta_A \circ \rho_\lambda((a_{k,i})_{i \in J}) - (a_{k,i})_{i \in J}\| + \|\Delta_A \circ \rho_\lambda((a_{k,i})_{i \notin J}) - (a_{k,i})_{i \notin J}\|$$

$$= \sum_{i \in J} \|\Delta_{A_i} \circ \rho_{i,\lambda}(a_{k,i}) - a_{k,i}\| + \sum_{i \notin J} \|a_{k,i}\|$$

$$< 2\epsilon,$$

and

$$\|\rho_\lambda(a_k b_k) - a_k \rho_\lambda(b_k)\| = \|\rho_\lambda((a_{k,i} b_{k,i})) - (a_{k,i})\rho_\lambda((b_{k,i}))\|$$
$$= \|\rho_\lambda((a_{k,i} b_{k,i}))_{i \in J} - (a_{k,i})_{i \in J}\rho_\lambda((b_{k,i})_{i \in J})\|$$
$$= \sum_{i \in J} \|\rho_{i,\lambda}(a_{k,i} b_{k,i}) - (a_{k,i})\rho_{i,\lambda}(b_{k,i})\|$$
$$< \epsilon.$$

Similarly, $\|\rho_\lambda(a_k b_k) - \rho_\lambda(a_k)b_k\| < \epsilon$. Therefore, A is approximately biprojective.

(ii) An inspection of the proof of part (i) shows that $\sup_\lambda \|\rho_\lambda\| \leq C$ if and only if $\sup_{i,\lambda} \|\rho_{i,\lambda}\| \leq C$. $\qquad\square$

As an application, we characterize (bounded) approximate biprojectivity of the Fourier algebra of a Clifford semigroup. We recall that a discrete group G is termed *amenable* if there is a positive linear functional $m \in l^\infty(G)^*$, called an *invariant mean*, such that $m(L_x\phi) = m(\phi)$ for all $x \in G$ and $\phi \in l^\infty(G)$, where $L_x\phi(y) = \phi(xy)$ for each $y \in G$.

Corollary 2.2. *Let S be a Clifford semigroup and $A(S)$ be the Fourier algebra of S, introduced in [6]. Then*

   (i) *$A(S)$ is approximately biprojective if and only if it admits an approximate identity.*
   (ii) *$A(S)$ is C-approximately biprojective if and only if $A(G_e)$ is C-approximately biprojective for each $e \in E(S)$.*
   (iii) *If G_e is amenable for each $e \in E(S)$, then $A(S)$ is C-approximately biprojective if and only if each G_e has an abelian subgroup of finite index and $\mathrm{BC}(A(G_e)) \leq C$.*

Proof. (i) By [6] we know that $A(S) = l^1 - \oplus_{e \in E(S)} A(G_e)$. Now the corollary follows from Proposition 2.1 and [7, Example 4.4 (ii)]. Note that $l^1 - \oplus_{e \in E(S)} A(G_e)$ has an approximate identity if and only if each $A(G_e)$ has an approximate identity.

(ii) This follows from Proposition 2.1 by noting that $A(S) = l^1 - \oplus_{e \in E(S)} A(G_e)$.

(iii) It follows from [7, Corollary 3.10 and Example 4.6 (ii)] and Proposition 2.1. $\qquad\square$

We remark that $A(\mathbb{F}_2)$ has an approximate identity (not bounded), where $\mathbb{F}_2$ is the free group on two generators, and so $A(\mathbb{F}_2)$ is approximately biprojective, but it is not biprojective [9].

Proposition 2.3. *Let A, B be two Banach algebras. If A is C-approximately biprojective and B is C'-approximately biprojective, then $A\hat{\otimes}B$ is CC'-approximately biprojective.*

Proof. Following the proof of [8, Proposition 2.4] we define $\rho_{(\alpha,\beta)} = \theta \circ (\rho_\alpha \otimes \tau_\beta)$ where

$$\theta : (A\hat{\otimes}A)\hat{\otimes}(B\hat{\otimes}B) \to (A\hat{\otimes}B)\hat{\otimes}(A\hat{\otimes}B)$$

is the canonical isometric isomorphism determined by

$$\theta((a_1 \otimes a_2) \otimes (b_1 \otimes b_2)) = (a_1 \otimes b_1) \otimes (a_2 \otimes b_2),$$

and (ρ_α) and (τ_β) are bounded by C and C' and satisfy Definition 1.1 for A and B, respectively. Then by the product order,

$$(\alpha,\beta) \leq (\alpha',\beta') \iff \alpha \leq \alpha' \text{ and } \beta \leq \beta',$$

we have

$$(2.1) \qquad \lim_{(\alpha,\beta)}\Delta_{A\hat{\otimes}B} \circ \rho_{(\alpha,\beta)}(a \otimes b) = \lim_{(\alpha,\beta)}\Delta_A \circ \rho_\alpha(a) \otimes \Delta_B \circ \tau_\beta(b) = a \otimes b.$$

Now let $F = \sum_{n=1}^\infty a_n \otimes b_n \in A\hat{\otimes}B$ and $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\sum_{n=N+1}^\infty \|a_n\|\|b_n\| < \epsilon$. Then by (2.1) there is (α_0, β_0) such that

$$\|\Delta_{A\hat{\otimes}B} \circ \rho_{(\alpha_0,\beta_0)}(\sum_{n=1}^N a_n \otimes b_n) - \sum_{n=1}^N a_n \otimes b_n\| < \epsilon,$$

and so

$$\|\Delta_{A\hat{\otimes}B} \circ \rho_{(\alpha_0,\beta_0)}(F) - F\| < \epsilon + \|\Delta_{A\hat{\otimes}B} \circ \rho_{(\alpha_0,\beta_0)}(\sum_{n=N+1}^\infty a_n \otimes b_n) - \sum_{n=N+1}^\infty a_n \otimes b_n\| < 2\epsilon + CC'\epsilon.$$

Also, since ρ_α's and τ_β's are bounded,

$$\lim_{(\alpha,\beta)} \rho_{(\alpha,\beta)}((a \otimes b)(a' \otimes b')) - (a \otimes b)\rho_{(\alpha,\beta)}(a' \otimes b')$$
$$= \lim_{(\alpha,\beta)}\theta[\rho_\alpha(aa') \otimes \tau_\beta(bb') - a\rho_\alpha(a') \otimes \tau_\beta(bb') + a\rho_\alpha(a') \otimes \tau_\beta(bb') - a\rho_\alpha(a') \otimes b\tau_\beta(b')]$$
$$= \lim_{(\alpha,\beta)}\theta[(\rho_\alpha(aa') - a\rho_\alpha(a')) \otimes \tau_\beta(bb') + a\rho_\alpha(a') \otimes (\tau_\beta(bb') - b\tau_\beta(b'))]$$
$$= 0.$$

Now let $F = \sum_{n=1}^\infty a_n \otimes b_n$, $G = \sum_{n=1}^\infty a'_n \otimes b'_n \in A\hat{\otimes}B$ and let $\epsilon > 0$. Then there is $M \in \mathbb{N}$ such that

$$\sum_{n=M+1}^\infty \|a_n\|\|b_n\| < \epsilon/\|G\|CC', \qquad \sum_{n=M+1}^\infty \|a'_n\|\|b'_n\| < \epsilon/\|F\|CC'.$$

If $F_M = \sum_{n=1}^M a_n \otimes b_n$ and $G_M = \sum_{n=1}^M a'_n \otimes b'_n$, then there is (α_1, β_1) such that

$$\|\rho_{(\alpha_1,\beta_1)}(F_M \cdot G_M) - F_M \cdot \rho_{(\alpha_1,\beta_1)}(G_M)\| < \epsilon,$$

and hence

$$\|\rho_{(\alpha_1,\beta_1)}(F \cdot G) - F \cdot \rho_{(\alpha_1,\beta_1)}(G)\| \leq \|\rho_{(\alpha_1,\beta_1)}(F_M \cdot G_M) - F_M \cdot \rho_{(\alpha_1,\beta_1)}(G_M)\|$$
$$+\|\rho_{(\alpha_1,\beta_1)}((F - F_M) \cdot G_M) - (F - F_M) \cdot \rho_{(\alpha_1,\beta_1)}(G_M)\|$$
$$+\|\rho_{(\alpha_1,\beta_1)}((F - F_M) \cdot (G - G_M)) - (F - F_M) \cdot \rho_{(\alpha_1,\beta_1)}(G - G_M)\|$$
$$+\|\rho_{(\alpha_1,\beta_1)}(F_M \cdot (G - G_M)) - F_M \cdot \rho_{(\alpha_1,\beta_1)}(G - G_M)\|$$
$$< \epsilon + 2CC'\tfrac{\epsilon}{\|G\|CC'}\|G_M\| + 2CC'\tfrac{\epsilon}{\|F\|CC'}\|F_M\| + 2CC'\tfrac{\epsilon^2}{\|F\|\|G\|C^2C'^2}$$
$$< 5\epsilon + \tfrac{2\epsilon}{CC'}.$$

Similarly, $\|\rho_{(\alpha_2,\beta_2)}(F \cdot G) - \rho_{(\alpha_2,\beta_2)}(F) \cdot G\| < L\epsilon$ for some $L > 0$ and some (α_2, β_2). $\qquad\square$

Proposition 2.4. *Let A be a unital Banach algebra and B a Banach algebra containing a non-zero idempotent. If $A\hat{\otimes}B$ is (boundedly) approximately biprojective, then so is A.*

Proof. Is the same as that of Proposition 2.6 of [8]. $\qquad\square$

Let A be a Banach algebra and let I be an arbitrary non-empty set. We denote by $\mathcal{M}_I(A)$ the space of all $I \times I$ matrices $(a_{i,j})_{i,j \in I}$ with entries in A such that $\sum_{i,j \in I} \|a_{i,j}\| < \infty$. Then $\mathcal{M}_I(A)$ becomes a Banach algebra under the usual matrix multiplication and the l^1-norm.

Proposition 2.5. *If A is a unital Banach algebra and I is an arbitrary non-empty set.*

(i) *$\mathcal{M}_I(A)$ is boundedly approximately biprojective if and only if A is boundedly approximately biprojective.*

(ii) *If $\mathcal{M}_I(A)$ is approximately biprojective, then so is A.*

Proof. Since $\mathcal{M}_I(A) \cong A \hat{\otimes} \mathcal{M}_I(\mathbb{C})$, part (i) follows from Propositions 2.3 and 2.4, and the fact that $\mathcal{M}_I(\mathbb{C})$ is 1-biprojective by [8, Proposition 2.7]. Part (ii) follows from Proposition 2.4, since A is unital and $\mathcal{M}_I(\mathbb{C})$ has a non-zero idempotent. $\square$

Let S be an inverse semigroup. There is a canonical partial order on $E(S)$ as follows:

$$p \leq q \Longleftrightarrow p = qp = pq \qquad (p, q \in E(S)).$$

If $(\mathcal{P}, \preceq)$ is a partially ordered set, we let $(p]_{\mathcal{P}} = \{q \in \mathcal{P} \mid q \preceq p\}$. Following [2], we say that $\mathcal{P}$ is *uniformly locally finite* if there exists $C > 0$ such that $|(p]_{\mathcal{P}}| \leq C$ for each $p \in \mathcal{P}$.

An inverse semigroup S is called *uniformly locally finite* if the partially ordered set $(E(S), \leq)$ has the corresponding property ([8, Definition 2.13]).

If S is a uniformly locally finite inverse semigroup, by [8, Theorem 2.18], we have

$$l^1(S) \cong l^1 - \bigoplus_{\lambda \in \Lambda} \mathcal{M}_{E(D_\lambda)}(l^1(G_{p_\lambda})),$$

where D_λ is a $\mathcal{D}$-class defined by the equivalence relation

$$s\mathcal{D}t \Longleftrightarrow \text{there exists } x \in S \text{ such that } Ss \cup \{s\} = Sx \cup \{x\} \text{ and } tS \cup \{t\} = xS \cup \{x\},$$

$E(D_\lambda)$ is the set of idempotents of D_λ, $p_\lambda \in E(D_\Lambda)$. Also, $G_{p_\lambda} = \{s \in S : ss^* = s^*s = p_\lambda\}$ is a maximal subgroup of S.

Theorem 2.6. *Let S be a uniformly locally finite inverse semigroup. Then*

(i) *$l^1(S)$ is approximately biprojective if and only if each G_{p_λ} is amenable.*

(ii) *$l^1(S)$ is C-approximately biprojective if and only if $BC(l^1(G_{p_\lambda})) \leq C$ and G_{p_λ} is amenable for each $\lambda \in \Lambda$.*

Proof. (i) Since $l^1(S) \cong l^1 - \bigoplus_{\lambda \in \Lambda} \mathcal{M}_{E(D_\lambda)}(l^1(G_{p_\lambda}))$, by Proposition 2.1, $l^1(S)$ is approximately biprojective if and only if $\mathcal{M}_{E(D_\lambda)}(l^1(G_{p_\lambda}))$ is approximately biprojective for each $\lambda \in \Lambda$. In view of Proposition 2.5, the latter implies the approximate biprojectivity of $l^1(G_{p_\lambda})$ for each $\lambda \in \Lambda$. Since $l^1(G_{p_\lambda})$ is unital, by [7, Corollary 3.7], this is equivalent to the approximate amenability of $l^1(G_{p_\lambda})$, hence to the amenability of G_{p_λ} for each $\lambda \in \Lambda$ ([4, Theorem 3.2]). Conversely, if G_{p_λ} is amenable, then $l^1(G_{p_\lambda})$ is amenable and so it is biflat. By [8, Proposition 2.7], $\mathcal{M}_{E(D_\lambda)}(l^1(G_{p_\lambda}))$ is biflat too. Now by [7, Theorem 3.3] we conclude that $\mathcal{M}_{E(D_\lambda)}(l^1(G_{p_\lambda}))$ is approximately biprojective. Therefore, by Proposition 2.1, $l^1(S)$ is approximately biprojective.

(ii) The proof of this part is similar to that of part (i).   □

Let G be a group with identity e, and let I be a non-empty set. Then the *Brandt inverse semigroup corresponding to G and I*, denoted $S = \mathcal{M}(G, I)$, is the collection of all $I \times I$ matrices $(g)_{ij}$ with $g \in G$ in the $(i, j)^{\text{th}}$ place and 0 (zero) elsewhere and the $I \times I$ zero matrix 0. Multiplication in S is given by the formula

$$(g)_{ij}(h)_{kl} = \begin{cases} (gh)_{il} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \qquad (g, h \in G,\ i, j, k, l \in I),$$

and $(g)_{ij}{}^* = (g^{-1})_{ji}$ and $0^* = 0$. The set of all idempotents is $E(S) = \{(e)_{ii} : i \in I\} \bigcup \{0\}$, and $G_{(e)_{ii}} \cong G$. It can be easily seen that each Brandt inverse semigroup is uniformly locally finite.

Corollary 2.7. *Let $S = \mathcal{M}(G, I)$ be the Brandt semigroup corresponding to G and I. Then $l^1(S)$ is (boundedly) approximately biprojective if and only if G is amenable.*

We remark that since $l^1(S)$ has the bounded approximation property, as we mentioned in the Introduction, the bounded approximate biprojectivity of $l^1(S)$ is equivalent to its biflatness. So the bounded case of Theorem 2.6 and Corollary 2.7 were obtained in [8].

3. **Approximate biprojectivity of some algebras over topological Brandt semigroups**

In this section, we study biprojectivity, biflatness and approximate biprojectivity of some Banach algebras related to topological Brandt semigroups. First we recall an analogue of measure algebra and group algebra for a topological semigroup. For a locally compact Hausdorff topological semigroup S, let $M(S)$ be the Banach space of all bounded complex-valued Radon measures on S with the total variation norm. With the convolution product $M(S)$ is a Banach algebra. We also set

$$M_a^l(S) = \{\mu \in M(S) \,|\, s \mapsto \delta_s * \mu \ \text{ is weakly continuous}\},$$

$$M_a^r(S) = \{\mu \in M(S) \,|\, s \mapsto \mu * \delta_s \ \text{ is weakly continuous}\},$$

$$M_a(S) = M_a^l(S) \cap M_a^r(S),$$

where δ_s denotes the Dirac measure at $s \in S$. It is well known that $M_a^l(S)$ and $M_a^r(S)$ are two sided L-ideals of $M(S)$ (see Lemma 3.5 and proof of Theorem 3.6 in [11]). It should be noted that in the case, where $S = G$ is a locally compact group, then $M_a(G) = M_a^l(G) = M_a^r(G) = L^1(G)$.

From now on, we take S to be $\mathcal{M}(G, I)$, a locally compact Hausdorff topological Brandt semigroup. For $A \subseteq G$ and $i, j \in I$ the set $\{(g)_{ij} : g \in A\}$ is denoted by $A_{i,j}$. In the case where $A = G$, the sets $G_{i,j}$ are open and homeomorphic, but not subsemigroups of S, unless $i = j$. Indeed, for each $i \in I$, $G_{i,i}$ is a maximal subgroup of S and also a locally compact topological group. Note that $G_{i,i}$'s are topologically isomorphic. Now fix $i_0 \in I$ and set

$$\tau_{(G,S)} = \{U \subseteq G : \ U_{i_0,i_0} \text{ is an open subset of } G_{i_0,i_0}\}.$$

Then $\tau_{(G,S)}$ is a topology on G turning it to a locally compact group. Notice that all subsets $G_{i,j}$ are homeomorphic to $(G, \tau_{(G,S)})$ and maximal subgroups $G_{i,i}$ are isomorphic to $(G, \tau_{(G,S)})$ as locally compact groups.

For $i, j \in I$, $\mu \in M(S)$ and $\nu \in M(G)$, we define

$$\mu_{i,j}(A) = \mu(A_{i,j}) \quad \text{and} \quad \nu^{i,j}(B) = \nu(B^{i,j}),$$

where A is a Borel subset of G, B is a Borel subset of S and $B^{i,j} = \{g \in G : (g)_{i,j} \in B\}$. We need the following two propositions ([10, Propositions 3 and 5]).

Proposition 3.1. *Let $S = \mathcal{M}(G, I)$ be a locally compact Hausdorff topological Brandt semigroup and G be equipped with the topology $\tau_{(G,S)}$.*

  (i) *For each $i, j \in I$ and $\mu \in M(G)$, $\mu^{i,j}$ is a well-defined measure in $M(S)$ such that $|\mu^{i,j}| = |\mu|^{i,j}$, $\|\mu^{i,j}\| = \|\mu\|$, and $\mathrm{supp}(\mu^{i,j}) = (\mathrm{supp}\,\mu)_{i,j}$.*

  (ii) *If $\mu, \nu \in M(G)$ and $i, j, k, l \in I$, then $\mu^{i,j} * \nu^{k,l} = (\mu * \nu)^{i,l}$ for $j = k$, and $\mu^{i,j} * \nu^{k,l} = \mu(G)\nu(G)\delta_0$ for $j \neq k$. Also, $\delta_0 * \nu^{k,l} = \nu^{k,l} * \delta_0 = \nu(G)\delta_0$.*

  (iii) *Let $\mu \in L^1(G)$ and $i, j \in I$. If for some $l_0 \in I$, $\bigcup_{i \in I} G_{i,l_0} (\bigcup_{i \in I} G_{l_0,i}$, respectively) is closed, then $\mu^{i,j} \in M_a^l(S)(\mu^{i,j} \in M_a^r(S)$, respectively).*

  (iv) *For each $i, j \in I$ and $\mu \in M(S)$, $\mu_{i,j}$ is a well-defined measure in $M(G)$ and $|\mu_{i,j}| = |\mu|_{i,j}$.*

  (v) *$\delta_0 \in M_a(S)$.*

  (vi) *If $\mu \in M_a^l(S)(\mu \in M_a^r(S)$, respectively) and $i, j \in I$, then $\mu_{i,j}^{i,j} \in M_a^l(S)$ ($\mu_{i,j}^{i,j} \in M_a^r(S)$, respectively), and $\mu_{i,j} \in L^1(G)$.*

Proposition 3.2. *Let $S = \mathcal{M}(G, I)$ be a locally compact Hausdorff topological Brandt semigroup and G be equipped with the topology $\tau_{(G,S)}$. Then*

$$M(S) = \left\{ \sum_{i,j \in I} \mu_{i,j}^{i,j} + c_0 \delta_0 : [\mu_{i,j}]_{i,j} \in \mathcal{M}_I(M(G)), c_0 \in \mathbb{C} \right\}$$

Moreover, if for some $l_0 \in I$, $\bigcup_{i \in I} G_{i,l_0}(\bigcup_{i \in I} G_{l_0,i}$, respectively) is closed, then $M_a^l(S)$ ($M_a^r(S)$, respectively) coincides with

$$\left\{ \sum_{i,j \in I} \mu_{i,j}^{i,j} + c_0 \delta_0 : [\mu_{i,j}]_{i,j} \in \mathcal{M}_I(L^1(G)), c_0 \in \mathbb{C} \right\},$$

otherwise $M_a^l(S) = \mathbb{C}\delta_0$ ($M_a^r(S) = \mathbb{C}\delta_0$, respectively), and $M_a(S) = \mathbb{C}\delta_0$.

Using above propositions, in the following theorem, we characterize $M(S)$ as a direct sum of $\mathcal{M}_I(M(G))$ and $\mathbb{C}$. The idea is taken from [10, Proposition 4].

Theorem 3.3. *Let $\Theta : \mathcal{M}_I(M(G)) \oplus_{l^1} \mathbb{C} \to M(S)$ be defined by*

$$\Theta([\mu_{i,j}]_{i,j}, c) = \sum_{i,j \in I} \mu_{i,j}^{i,j} - \sum_{i,j \in I} \mu_{i,j}(G)\delta_0 + c\delta_0.$$

Then Θ is a Banach algebra isomorphism.

Proof. Define the maps $\Phi : \mathcal{M}_I(M(G)) \to M(S)$ and $\Psi : M(S) \to \mathbb{C}$ by

$$\Phi([\mu_{i,j}]_{i,j}) = \sum_{i,j \in I} \mu_{i,j}^{i,j} - \sum_{i,j \in I} \mu_{i,j}(G)\delta_0 \quad \text{and} \quad \Psi(\mu) = \mu(S).$$

Clearly, Ψ is a continuous epimorphism. To see that Φ is a homomorphism, let $[\mu_{i,j}]_{i,j}, [\nu_{i,j}]_{i,j} \in \mathcal{M}_I(M(G))$. Then

$$
\begin{aligned}
\Phi([\mu_{i,j}]_{i,j}) * \Phi([\nu_{i,j}]_{i,j}) &= \Big(\sum_{i,l \in I} \mu_{i,l}^{i,l} - \sum_{i,l \in I} \mu_{i,l}(G)\delta_0 \Big)\Big(\sum_{k,j \in I} \nu_{k,j}^{k,j} - \sum_{k,j \in I} \nu_{k,j}(G)\delta_0 \Big) \\
&= \sum_{i,l \in I}\sum_{k,j \in I} \mu_{i,l}^{i,l} * \nu_{k,j}^{k,j} - \Big(\sum_{i,l \in I} \mu_{i,l}(G) \Big)\Big(\sum_{k,j \in I} \nu_{k,j}(G) \Big)\delta_0 \\
&\quad - \Big(\sum_{i,l \in I} \mu_{i,l}(G) \Big)\Big(\sum_{k,j \in I} \nu_{k,j}(G) \Big)\delta_0 + \Big(\sum_{i,l \in I} \mu_{i,l}(G) \Big)\Big(\sum_{k,j \in I} \nu_{k,j}(G) \Big)\delta_0 * \delta_0 \\
&= \sum_{i,l \in I}\sum_{k,j \in I} \mu_{i,l}^{i,l} * \nu_{k,j}^{k,j} - \Big(\sum_{i,l \in I} \mu_{i,l}(G) \Big)\Big(\sum_{k,j \in I} \nu_{k,j}(G) \Big)\delta_0 \\
&= \sum_{i,j,k \in I} \mu_{i,k}^{i,k} * \nu_{k,j}^{k,j} + \sum_{i \neq k, i,j,k,l \in I} \mu_{i,l}^{i,l} * \nu_{k,j}^{k,j} - \sum_{i,j,k,l \in I} \mu_{i,l}(G)\nu_{k,j}(G)\delta_0 \\
&= \sum_{i,j,k \in I} \mu_{i,k}^{i,k} * \nu_{k,j}^{k,j} + \sum_{k \neq l, i,j,k,l \in I} \mu_{i,l}(G)\nu_{k,j}(G)\delta_0 - \sum_{i,j,k,l \in I} \mu_{i,l}(G)\nu_{k,j}(G)\delta_0 \\
&= \sum_{i,j,k \in I} \mu_{i,k}^{i,k} * \nu_{k,j}^{k,j} - \sum_{k = l, j,k,l \in I} \mu_{i,l}(G)\nu_{k,j}(G)\delta_0 \\
&= \sum_{i,j \in I}\sum_{k \in I} \mu_{i,k}^{i,k} * \nu_{k,j}^{k,j} - \sum_{i,j \in I}\sum_{k \in I} \mu_{i,k}(G)\nu_{k,j}(G)\delta_0 \\
&= \sum_{i,j \in I}\sum_{k \in I} (\mu_{i,k} * \nu_{k,j})^{i,j} - \Big(\sum_{i,j \in I}(\sum_{k \in I} \mu_{i,k} * \nu_{k,j})(G) \Big)\delta_0 \\
&= \Phi([\sum_{k \in I} \mu_{i,j} * \nu_{k,j}]_{i,j}) \\
&= \Phi([\mu_{i,j}]_{i,j}[\nu_{i,j}]_{i,j}).
\end{aligned}
$$

By noting that $S^{i,j} = G$ for any $i, j \in I$, we have

$$\Psi\Phi([\mu_{i,j}]_{i,j}) = \sum_{i,j \in I} \mu_{i,j}^{i,j}(S) - \sum_{i,j \in I} \mu_{i,j}(G) = \sum_{i,j \in I} \mu_{i,j}(G) - \sum_{i,j \in I} \mu_{i,j}(G) = 0,$$

which shows that $\mathrm{Im}(\Phi) \subseteq \ker(\Psi)$. For the converse inclusion, let $\mu \in \ker \Psi$. By Proposition 3.2, μ is of the form $\mu = \sum_{i,j \in I} \mu_{i,j}^{i,j} + c_0\delta_0$, where $[\mu_{i,j}]_{i,j} \in \mathcal{M}_I(M(G))$ and $c_0 \in \mathbb{C}$. Since

$$0 = \mu(S) = \sum_{i,j \in I} \mu_{i,j}^{i,j}(S) + c_0 = \sum_{i,j \in I} \mu_{i,j}(G) + c_0,$$

we have

$$\mu = \sum_{i,j \in I} \mu_{i,j}^{i,j} - \sum_{i,j \in I} \mu_{i,j}(G)\delta_0 = \Phi([\mu_{i,j}]_{i,j}) \in \mathrm{Im}(\Phi).$$

Therefore, $\mathrm{Im}(\Phi) = \ker(\Psi)$ and so $\mathrm{Im}(\Phi)$ is closed in $M(S)$. To see that Φ is injective, let $[\mu_{i,j}]_{i,j} \in \ker(\Phi)$. Then $\sum_{i,j \in I} \mu_{i,j}^{i,j} - \sum_{i,j \in I} \mu_{i,j}(G)\delta_0 = 0$. Thus, for any Borel set $E \subseteq G$ and any $i_0, j_0 \in I$,

$$(3.1) \qquad 0 = (\sum_{i,j \in I} \mu_{i,j}^{i,j} - \sum_{i,j \in I} \mu_{i,j}(G)\delta_0)(E^{i_0,j_0}) = \mu_{i_0,j_0}^{i_0,j_0}(E^{i_0,j_0}) = \mu_{i_0,j_0}(E).$$

Therefore, $[\mu_{i,j}]_{i,j} = 0$. A direct calculation shows that

$$\mathrm{Im}(\Phi) \subseteq \{\mu \in M(S) : \mu * \delta_0 = \delta_0 * \mu = 0\}.$$

This together with $\delta_0^2 = \delta_0$ imply that Θ is a homomorphism. Indeed,

$$
\begin{aligned}
\Theta([\mu_{i,j}]_{i,j}, c) * \Theta([\nu_{i,j}]_{i,j}, d) &= (\Phi([\mu_{i,j}]_{i,j}) + c\delta_0) * (\Phi([\nu_{i,j}]_{i,j}) + d\delta_0) \\
&= \Phi([\mu_{i,j}]_{i,j})(\Phi([\nu_{i,j}]_{i,j}) + cd\delta_0 \\
&= \Phi([\mu_{i,j}]_{i,j}[\nu_{i,j}]_{i,j}) + cd\delta_0 \\
&= \Theta([\mu_{i,j}]_{i,j}[\nu_{i,j}]_{i,j}, cd) \\
&= \Theta(([\mu_{i,j}]_{i,j}, c)([\nu_{i,j}]_{i,j}, d)).
\end{aligned}
$$

Since $\delta_0 \in \mathrm{Im}(\Theta) \backslash \mathrm{Im}(\Phi)$, we have $\mathrm{Im}(\Phi) \subsetneq \mathrm{Im}(\Theta)$. Since $\dim(\frac{M(S)}{\mathrm{Im}(\Phi)}) = 1$, it follows that $\mathrm{Im}(\Theta) = M(S)$. Also, it is clear that Θ is injective. Therefore, Θ is an isomorphism of Banach algebras.   □

Corollary 3.4. *Let $S = \mathcal{M}(G, I)$ be a locally compact Hausdorff topological Brandt semigroup and G be equipped with the topology $\tau_{(G,S)}$ in such a way that for some $l_0 \in I$, $\bigcup_{i \in I} G_{i,l_0}(\bigcup_{i \in I} G_{l_0,i},$ respectively) is closed. Then the isomorphism $M(S) \cong \mathcal{M}_I(M(G)) \oplus_{l^1} \mathbb{C}$ induces the Banach algebra isomorphism*

$$M_a^l(S) \cong \mathcal{M}_I(L^1(G)) \oplus_{l^1} \mathbb{C} \quad (M_a^r(S) \cong \mathcal{M}_I(L^1(G)) \oplus_{l^1} \mathbb{C}).$$

Theorem 3.5. *Let $S = \mathcal{M}(G, I)$ be a locally compact Hausdorff topological Brandt semigroup and G be equipped with the topology $\tau_{(G,S)}$. The following are equivalent:*

  (i) *$M(S)$ is boundedly approximately biprojective.*
  (ii) *$M(S)$ is approximately biprojective.*
  (iii) *G is discrete and amenable.*

Proof. (i) $\implies$ (ii) is trivial.

(ii) $\implies$ (iii): Since $M(S) \cong \mathcal{M}_I(M(G)) \oplus_{l^1} \mathbb{C}$ as Banach algebras, by Propositions 2.1 and 2.5, approximate biprojectivity of $M(S)$ implies the approximate biprojectivity of $M(G)$. This together with [7, Example 4.5] show that G is discrete and amenable.

(iii) $\implies$ (i): If G is discrete and amenable, then $M(G)$ is amenable and so bounded approximately biprojective by [7, Corollary 3.10]. Now it follows from Propositions 2.5 and 2.1 and the isomorphism $M(S) \cong \mathcal{M}_I(M(G)) \oplus_{l^1} \mathbb{C}$ that $M(S)$ is boundedly approximately biprojective.   □

The following corollary is the topological version of [8, Theorem 3.7], when S is a Brandt semigroup.

Corollary 3.6. *Let $S = \mathcal{M}(G, I)$ be a locally compact Hausdorff topological Brandt semigroup and G be equipped with the topology $\tau_{(G,S)}$. Then $M(S)$ is biflat if and only if G is discrete and amenable.*

Proof. Since $M(S)$ has the bounded approximation property, the corollary follows from Theorem 3.5, [1, Theorem 3.6(A)] and [7, Theorem 3.3].   □

Applying [8, Proposition 2.7], one can easily observe that $M(S)$ is biprojective if and only if G is finite. The next corollary is a generalization of [3, Theorem 1.3], when S is a Brandt semigroup.

Corollary 3.7. *Let $S = \mathcal{M}(G, I)$ be a locally compact Hausdorff topological Brandt semigroup and G be equipped with the topology $\tau_{(G,S)}$. Then $M(S)$ is amenable if and only if I is finite and G is discrete and amenable.*

Proof. The isomorphism $M(S) \cong \mathcal{M}_I(M(G)) \oplus_{l^1} \mathbb{C}$ shows that $M(S)$ has a bounded approximate identity if and only if I is finite. Now Theorem 3.5 and [7, Corollary 3.10] complete the proof.   □

For a locally compact group G the condition $M(G) = l^1(G)$ implies the discreteness of G, but this phenomenon does not hold for a locally compact semigroup S. Thus the discreteness of G in Theorem 3.5 does not necessarily induce the discreteness of S. Nonetheless, we have the following corollary by applying [10, Corollary 2].

Corollary 3.8. *Let $S = \mathcal{M}(G, I)$ be a locally compact Hausdorff topological Brandt semigroup and G be equipped with the topology $\tau_{(G,S)}$. Let $M(S)$ be approximately biprojective. Then S is a fundamental semigroup and $M(S) = M_a(S) = l^1(S)$.*

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