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APPROXIMATE BIPROJECTIVITY AND BIFLATNESS OF SOME ALGEBRAS OVER CERTAIN SEMIGROUPS

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ABSTRACT. We investigate (bounded) approximate biprojectivity of $l^1(S)$ for uniformly locally finite inverse semigroups. As a consequence, we show that when $S = \mathcal{M}(G, I)$ is the Brandt inverse semigroup, then $l^1(S)$ is (boundedly) approximately biprojective if and only if G is amenable. Moreover, we study biflatness and (bounded) approximate biprojectivity of the measure algebra $M(S)$ of a topological Brandt semigroup.

1. **Introduction**

In [8], Ramsden has characterized biprojectivity and biflatness of the semigroup algebra $l^1(S)$ for an inverse semigroup S. Motivated by this investigation, in this paper, we verify (bounded) approximate biprojectivity of $l^1(S)$ when S is a uniformly locally finite inverse semigroup. Then we characterize (bounded) approximate biprojectivity of $l^1(S)$ for the Brandt inverse semigroup $S = \mathcal{M}(G, I)$. We also characterize (bounded) approximate biprojectivity of the Fourier algebra $A(S)$ of a Clifford semigroup S. In the last section, we study biflatness and approximate biprojectivity of some Banach algebras related to topological Brandt semigroups. First we recall briefly definitions.

A Banach space X is said to have the (*bounded*) *approximation property* if there exists a (bounded) net (T_α) of finite rank operators on X such that $T_\alpha \to I_X$ uniformly on compact subsets of X, where I_X is the identity map on X.

For two Banach spaces X and Y, we denote by $X \hat{\otimes} Y$ their projective tensor product and by $\mathcal{B}(X, Y)$ the space of all bounded linear operators from X to Y. Suppose X_1, X_2, Y_1 and Y_2 are

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Banach spaces and $T_i \in \mathcal{B}(X_i, Y_i)$ for $i = 1, 2$. The tensor product of T_1 and T_2 is the linear map defined by $T_1 \otimes T_2 : X_1 \hat{\otimes} X_2 \to Y_1 \hat{\otimes} Y_2$ by

$$(T_1 \otimes T_2)(x_1 \otimes x_2) = T_1(x_1) \otimes T_2(x_2) \qquad (x_1 \in X_1, x_2 \in X_2).$$

The projective tensor product of two Banach algebras A and B is denoted by $A \hat{\otimes} B$, which is a Banach algebra with the following multiplication:

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2) \qquad (a_1, a_2 \in A,\ b_1, b_2 \in B).$$

Let A be a Banach algebra. The product map on A extends to a map $\Delta_A : A \hat{\otimes} A \to A$, determined by $\Delta_A(a \otimes b) = ab$ for all $a, b \in A$. The projective tensor product $A \hat{\otimes} A$ becomes a Banach A-bimodule with the following module actions:

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \qquad (a, b, c \in A).$$

By these actions Δ_A becomes an A-bimodule homomorphism.

An *approximate diagonal* for A is a net $(\mathbf{m}_\alpha) \subseteq A \hat{\otimes} A$ such that $a \cdot \mathbf{m}_\alpha - \mathbf{m}_\alpha \cdot a \to 0$ and $(\Delta_A \mathbf{m}_\alpha) a \to a$ for each $a \in A$. A Banach algebra A is called *pseudo-amenable* if it has an approximate diagonal ([5]).

A Banach algebra A is called *biprojective* if Δ_A has a bounded right inverse which is an A-bimodule homomorphism and is called *biflat* if Δ_A^* has a bounded left inverse which is an A-bimodule homomorphism. Different notations of approximate biprojectivity are considered in [1,7] and [12]. We use the given one by the first author in [7] that includes much more classes of Banach algebras; see [7].

Definition 1.1. A Banach algebra A is called *approximately biprojective* if there exists a net $(\rho_\alpha) \subset \mathcal{B}(A, A \hat{\otimes} A)$ such that

(i) $\Delta_A \circ \rho_\alpha(a) \to a$ for all $a \in A$,
(ii) $a \cdot \rho_\alpha(b) - \rho_\alpha(ab) \to 0$ for all $a, b \in A$,
(iii) $\rho_\alpha(a) \cdot b - \rho_\alpha(ab) \to 0$ for all $a, b \in A$.

Moreover, if the net (ρ_α) can be chosen to be bounded with bound C, we say that A is *boundedly approximately biprojective* or *C-approximately biprojective*. The infimum of such $C > 0$ is called the *approximate biprojectivity constant* of A and is denoted by $\mathrm{BC}(A)$.

Notice that Zhang assumes each ρ_α in Definition 1.1 is an A-bimodule homomorphism and Aristov assumes the convergence in conditions (i), (ii) and (iii) is the uniformly convergence on compact subsets of A; [1, Proposition 5.5]. We remark that our definition coincides with Aristov's definition in bounded case; see [1, Theorem 7.2]. Note also that if A has the bounded approximation property, by [1, Theorems 3.6(B) and 7.3(C)], bounded approximate biprojectivity is equivalent to biflatness.

A (discrete) semigroup S is called an *inverse semigroup* if for each $s \in S$ there is a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. An element $e \in S$ is called an *idempotent* if $e^2 = e^* = e$. The set of all idempotents of S is denoted by $E(S)$. For each $e \in E(S)$ we define $G_e = \{s \in S : ss^* = s^*s = e\}$, which is a maximal subgroup of S. An inverse semigroup S is called a *Clifford semigroup* if $s^*s = ss^*$ for all $s \in S$.

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2. Approximate biprojectivity of some semigroup algebras

In the following proposition we characterize (bounded) approximate biprojectivity of l^1-direct sum of Banach algebras. Let $(A_i)_{i\in I}$ be a family of Banach algebras. Then the l^1-direct sum of $(A_i)_{i\in I}$ is defined by

$$A = l^1 - \bigoplus_{i\in I} A_i = \left\{ a = (a_i)_{i\in I} : \|a\| = \sum_{i\in I} \|a_i\|_{A_i} < \infty \right\}.$$

Obviously, A is a Banach algebra with componentwise product.

Proposition 2.1. *Let $(A_i)_{i\in I}$ be a family of Banach algebras and let $A = l^1 - \bigoplus_{i\in I} A_i$. Then*

(i) A is approximately biprojective if and only if each A_i is approximately biprojective.

(ii) A is C-approximately biprojective if and only if each A_i is C-approximately biprojective.

Proof. Let A be approximately biprojective and $(\rho_\lambda) \subset \mathcal{B}(A, A\hat{\otimes}A)$ be a net satisfying Definition 1.1. Let also for each $i \in I$, $\iota_i : A_i \to A$ and $p_i : A \to A_i$ be the canonical injection and projection, respectively. Set $\rho_{i,\lambda} = (p_i \otimes p_i) \circ \rho_\lambda \circ \iota_i \in \mathcal{B}(A_i, A_i\hat{\otimes}A_i)$. Then for each $a_i \in A_i$,

$$\lim_\lambda \Delta_{A_i} \circ \rho_{i,\lambda}(a_i) = \lim_\lambda \Delta_{A_i} \circ (p_i \otimes p_i) \circ \rho_\lambda \circ \iota_i(a_i) = \lim_\lambda p_i \circ \Delta_A \circ \rho_\lambda(\iota_i(a_i)) = p_i(\iota_i(a_i)) = a_i,$$

and

$$\lim_\lambda \rho_{i,\lambda}(a_i b_i) - a_i\rho_{i,\lambda}(b_i) = \lim_\lambda (p_i \otimes p_i)[\rho_\lambda(\iota_i(a_i)\iota_i(b_i)) - \iota_i(a_i)\rho_\lambda(\iota_i(b_i))] = 0,$$

and likewise, $\lim_\lambda \rho_{i,\lambda}(a_i b_i) - \rho_{i,\lambda}(a_i)b_i = 0$. So each A_i is approximately biprojective.

Conversely, assume that each A_i is approximately biprojective. Take $\epsilon > 0$, $F = \{a_1, \dots, a_n\} \subset A$ and $G = \{b_1, \dots, b_n\} \subset A$. Suppose $a_k = (a_{k,i})$ and $b_k = (b_{k,i})$ for $1 \le k \le n$. Then there is a finite subset J of I such that

$$\sum_{i\notin J} \|a_{k,i}\| < \epsilon \quad \text{and} \quad \sum_{i\notin J} \|b_{k,i}\| < \epsilon \qquad (1 \le k \le n).$$

Now for all $i \in J$ there exist $\lambda = \lambda(F, G, \epsilon)$ and $\rho_{i,\lambda} : A_i \to A_i\hat{\otimes}A_i$ such that

$$\|\Delta_{A_i} \circ \rho_{i,\lambda}(a_{k,i}) - a_{k,i}\| < \frac{\epsilon}{\#J},$$

and

$$\|\rho_{i,\lambda}(a_{k,i}b_{k,i}) - a_{k,i}\rho_{i,\lambda}(b_{k,i})\| < \frac{\epsilon}{\#J}, \quad \|\rho_{i,\lambda}(a_{k,i}b_{k,i}) - \rho_{i,\lambda}(a_{k,i})b_{k,i}\| < \frac{\epsilon}{\#J}.$$

Define $\rho_\lambda = \rho_{\lambda(F,G,\epsilon)} := l^1 - \oplus_{i\in J}\rho_{i,\lambda(F,G,\epsilon)}$. Then for each $1 \le k \le n$,

$$\begin{aligned} \|\Delta_A \circ \rho_\lambda(a_k) - a_k\| &\le \|\Delta_A \circ \rho_\lambda((a_{k,i})_{i\in J}) - (a_{k,i})_{i\in J}\| + \|\Delta_A \circ \rho_\lambda((a_{k,i})_{i\notin J}) - (a_{k,i})_{i\notin J}\| \\ &= \textstyle\sum_{i\in J} \|\Delta_{A_i} \circ \rho_{i,\lambda}(a_{k,i}) - a_{k,i}\| + \sum_{i\notin J} \|a_{k,i}\| \\ &< 2\epsilon, \end{aligned}$$

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and

$$\begin{aligned}\|\rho_\lambda(a_k b_k) - a_k\rho_\lambda(b_k)\| &= \|\rho_\lambda((a_{k,i}b_{k,i})) - (a_{k,i})\rho_\lambda((b_{k,i}))\| \\ &= \|\rho_\lambda((a_{k,i}b_{k,i}))_{i\in J}) - (a_{k,i})_{i\in J}\rho_\lambda((b_{k,i})_{i\in J})\| \\ &= \textstyle\sum_{i\in J}\|\rho_{i,\lambda}(a_{k,i}b_{k,i}) - (a_{k,i})\rho_{i,\lambda}(b_{k,i})\| \\ &< \epsilon.\end{aligned}$$

Similarly, $\|\rho_\lambda(a_k b_k) - \rho_\lambda(a_k)b_k\| < \epsilon$. Therefore, A is approximately biprojective.

(ii) An inspection of the proof of part (i) shows that $\sup_\lambda \|\rho_\lambda\| \leq C$ if and only if $\sup_{i,\lambda}\|\rho_{i,\lambda}\| \leq C$. □

As an application, we characterize (bounded) approximate biprojectivity of the Fourier algebra of a Clifford semigroup. We recall that a discrete group G is termed *amenable* if there is a positive linear functional $m \in l^\infty(G)^*$, called an *invariant mean*, such that $m(L_x\phi) = m(\phi)$ for all $x \in G$ and $\phi \in l^\infty(G)$, where $L_x\phi(y) = \phi(xy)$ for each $y \in G$.

Corollary 2.2. *Let S be a Clifford semigroup and $A(S)$ be the Fourier algebra of S, introduced in [6]. Then*

(i) *$A(S)$ is approximately biprojective if and only if it admits an approximate identity.*
(ii) *$A(S)$ is C-approximately biprojective if and only if $A(G_e)$ is C-approximately biprojective for each $e \in E(S)$.*
(iii) *If G_e is amenable for each $e \in E(S)$, then $A(S)$ is C-approximately biprojective if and only if each G_e has an abelian subgroup of finite index and $\mathrm{BC}(A(G_e)) \leq C$.*

Proof. (i) By [6] we know that $A(S) = l^1 - \oplus_{e\in E(S)}A(G_e)$. Now the corollary follows from Proposition 2.1 and [7, Example 4.4 (ii)]. Note that $l^1 - \oplus_{e\in E(S)}A(G_e)$ has an approximate identity if and only if each $A(G_e)$ has an approximate identity.

(ii) This follows from Proposition 2.1 by noting that $A(S) = l^1 - \oplus_{e\in E(S)}A(G_e)$.

(iii) It follows from [7, Corollary 3.10 and Example 4.6 (ii)] and Proposition 2.1. □

We remark that $A(\mathbb{F}_2)$ has an approximate identity (not bounded), where $\mathbb{F}_2$ is the free group on two generators, and so $A(\mathbb{F}_2)$ is approximately biprojective, but it is not biprojective [9].

Proposition 2.3. *Let A, B be two Banach algebras. If A is C-approximately biprojective and B is C'-approximately biprojective, then $A\hat{\otimes}B$ is CC'-approximately biprojective.*

Proof. Following the proof of [8, Proposition 2.4] we define $\rho_{(\alpha,\beta)} = \theta \circ (\rho_\alpha \otimes \tau_\beta)$ where

$$\theta : (A\hat{\otimes}A)\hat{\otimes}(B\hat{\otimes}B) \to (A\hat{\otimes}B)\hat{\otimes}(A\hat{\otimes}B)$$

is the canonical isometric isomorphism determined by

$$\theta((a_1 \otimes a_2) \otimes (b_1 \otimes b_2)) = (a_1 \otimes b_1) \otimes (a_2 \otimes b_2),$$

and (ρ_α) and (τ_β) are bounded by C and C' and satisfy Definition 1.1 for A and B, respectively. Then by the product order,

$$(\alpha,\beta) \leq (\alpha',\beta') \Longleftrightarrow \alpha \leq \alpha' \text{ and } \beta \leq \beta',$$

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we have

$$\lim_{(\alpha,\beta)}\Delta_{A\hat{\otimes}B}\circ\rho_{(\alpha,\beta)}(a\otimes b)=\lim_{(\alpha,\beta)}\Delta_A\circ\rho_\alpha(a)\otimes\Delta_B\circ\tau_\beta(b)=a\otimes b. \tag{2.1}$$

Now let $F=\sum_{n=1}^\infty a_n\otimes b_n\in A\hat{\otimes}B$ and $\epsilon>0$. Choose $N\in\mathbb{N}$ such that $\sum_{n=N+1}^\infty\|a_n\|\|b_n\|<\epsilon$. Then by (2.1) there is (α_0,β_0) such that

$$\|\Delta_{A\hat{\otimes}B}\circ\rho_{(\alpha_0,\beta_0)}(\sum_{n=1}^N a_n\otimes b_n)-\sum_{n=1}^N a_n\otimes b_n\|<\epsilon,$$

and so

$$\|\Delta_{A\hat{\otimes}B}\circ\rho_{(\alpha_0,\beta_0)}(F)-F\|<\epsilon+\|\Delta_{A\hat{\otimes}B}\circ\rho_{(\alpha_0,\beta_0)}(\sum_{n=N+1}^\infty a_n\otimes b_n)-\sum_{n=N+1}^\infty a_n\otimes b_n\|<2\epsilon+CC'\epsilon.$$

Also, since ρ_α's and τ_β's are bounded,

$$\begin{aligned}
\lim_{(\alpha,\beta)}&\rho_{(\alpha,\beta)}((a\otimes b)(a'\otimes b'))-(a\otimes b)\rho_{(\alpha,\beta)}(a'\otimes b')\\
&=\lim_{(\alpha,\beta)}\theta[\rho_\alpha(aa')\otimes\tau_\beta(bb')-a\rho_\alpha(a')\otimes\tau_\beta(bb')+a\rho_\alpha(a')\otimes\tau_\beta(bb')-a\rho_\alpha(a')\otimes b\tau_\beta(b')]\\
&=\lim_{(\alpha,\beta)}\theta[(\rho_\alpha(aa')-a\rho_\alpha(a'))\otimes\tau_\beta(bb')+a\rho_\alpha(a')\otimes(\tau_\beta(bb')-b\tau_\beta(b'))]\\
&=0.
\end{aligned}$$

Now let $F=\sum_{n=1}^\infty a_n\otimes b_n$, $G=\sum_{n=1}^\infty a'_n\otimes b'_n\in A\hat{\otimes}B$ and let $\epsilon>0$. Then there is $M\in\mathbb{N}$ such that

$$\sum_{n=M+1}^\infty\|a_n\|\|b_n\|<\epsilon/\|G\|CC',\qquad\sum_{n=M+1}^\infty\|a'_n\|\|b'_n\|<\epsilon/\|F\|CC'.$$

If $F_M=\sum_{n=1}^M a_n\otimes b_n$ and $G_M=\sum_{n=1}^M a'_n\otimes b'_n$, then there is (α_1,β_1) such that

$$\|\rho_{(\alpha_1,\beta_1)}(F_M\cdot G_M)-F_M\cdot\rho_{(\alpha_1,\beta_1)}(G_M)\|<\epsilon,$$

and hence

$$\begin{aligned}
\|\rho_{(\alpha_1,\beta_1)}(F\cdot G)\ &-F\cdot\rho_{(\alpha_1,\beta_1)}(G)\|\le\|\rho_{(\alpha_1,\beta_1)}(F_M\cdot G_M)-F_M\cdot\rho_{(\alpha_1,\beta_1)}(G_M)\|\\
&+\|\rho_{(\alpha_1,\beta_1)}((F-F_M)\cdot G_M)-(F-F_M)\cdot\rho_{(\alpha_1,\beta_1)}(G_M)\|\\
&+\|\rho_{(\alpha_1,\beta_1)}((F-F_M)\cdot(G-G_M))-(F-F_M)\cdot\rho_{(\alpha_1,\beta_1)}(G-G_M)\|\\
&+\|\rho_{(\alpha_1,\beta_1)}(F_M\cdot(G-G_M))-F_M\cdot\rho_{(\alpha_1,\beta_1)}(G-G_M)\|\\
&<\epsilon+2CC'\tfrac{\epsilon}{\|G\|CC'}\|G_M\|+2CC'\tfrac{\epsilon}{\|F\|CC'}\|F_M\|+2CC'\tfrac{\epsilon^2}{\|F\|\|G\|C^2C'^2}\\
&<5\epsilon+\tfrac{2\epsilon}{CC'}.
\end{aligned}$$

Similarly, $\|\rho_{(\alpha_2,\beta_2)}(F\cdot G)-\rho_{(\alpha_2,\beta_2)}(F)\cdot G\|<L\epsilon$ for some $L>0$ and some (α_2,β_2). □

Proposition 2.4. *Let A be a unital Banach algebra and B a Banach algebra containing a non-zero idempotent. If $A\hat{\otimes}B$ is (boundedly) approximately biprojective, then so is A.*

Proof. Is the same as that of Proposition 2.6 of [8]. □

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Let A be a Banach algebra and let I be an arbitrary non-empty set. We denote by $\mathcal{M}_I(A)$ the space of all $I \times I$ matrices $(a_{i,j})_{i,j\in I}$ with entries in A such that $\sum_{i,j\in I} \|a_{i,j}\| < \infty$. Then $\mathcal{M}_I(A)$ becomes a Banach algebra under the usual matrix multiplication and the l^1-norm.

Proposition 2.5. *If A is a unital Banach algebra and I is an arbitrary non-empty set.*

(i) *$\mathcal{M}_I(A)$ is boundedly approximately biprojective if and only if A is boundedly approximately biprojective.*

(ii) *If $\mathcal{M}_I(A)$ is approximately biprojective, then so is A.*

Proof. Since $\mathcal{M}_I(A) \cong A\hat{\otimes}\mathcal{M}_I(\mathbb{C})$, part (i) follows from Propositions 2.3 and 2.4, and the fact that $\mathcal{M}_I(\mathbb{C})$ is 1-biprojective by [8, Proposition 2.7]. Part (ii) follows from Proposition 2.4, since A is unital and $\mathcal{M}_I(\mathbb{C})$ has a non-zero idempotent. □

Let S be an inverse semigroup. There is a canonical partial order on $E(S)$ as follows:

$$p \leq q \Longleftrightarrow p = qp = pq \qquad\qquad (p, q \in E(S)).$$

If $(\mathcal{P}, \preceq)$ is a partially ordered set, we let $(p]_{\mathcal{P}} = \{q \in \mathcal{P} \,|\, q \preceq p\}$. Following [2], we say that $\mathcal{P}$ is *uniformly locally finite* if there exists $C > 0$ such that $|(p]_{\mathcal{P}}| \leq C$ for each $p \in \mathcal{P}$.

An inverse semigroup S is called *uniformly locally finite* if the partially ordered set $(E(S), \leq)$ has the corresponding property ([8, Definition 2.13]).

If S is a uniformly locally finite inverse semigroup, by [8, Theorem 2.18], we have

$$l^1(S) \cong l^1 - \bigoplus_{\lambda\in\Lambda} \mathcal{M}_{E(D_\lambda)}(l^1(G_{p_\lambda})),$$

where D_λ is a $\mathcal{D}$-class defined by the equivalence relation

$$s\mathcal{D}t \Longleftrightarrow \text{there exists } x \in S \text{ such that } Ss \cup \{s\} = Sx \cup \{x\} \text{ and } tS \cup \{t\} = xS \cup \{x\},$$

$E(D_\lambda)$ is the set of idempotents of D_λ, $p_\lambda \in E(D_\Lambda)$. Also, $G_{p_\lambda} = \{s \in S : ss^* = s^*s = p_\lambda\}$ is a maximal subgroup of S.

Theorem 2.6. *Let S be a uniformly locally finite inverse semigroup. Then*

(i) *$l^1(S)$ is approximately biprojective if and only if each G_{p_λ} is amenable.*

(ii) *$l^1(S)$ is C-approximately biprojective if and only if $BC(l^1(G_{p_\lambda})) \leq C$ and G_{p_λ} is amenable for each $\lambda \in \Lambda$.*

Proof. (i) Since $l^1(S) \cong l^1 - \bigoplus_{\lambda\in\Lambda} \mathcal{M}_{E(D_\lambda)}(l^1(G_{p_\lambda}))$, by Proposition 2.1, $l^1(S)$ is approximately biprojective if and only if $\mathcal{M}_{E(D_\lambda)}(l^1(G_{p_\lambda}))$ is approximately biprojective for each $\lambda \in \Lambda$. In view of Proposition 2.5, the latter implies the approximate biprojectivity of $l^1(G_{p_\lambda})$ for each $\lambda \in \Lambda$. Since $l^1(G_{p_\lambda})$ is unital, by [7, Corollary 3.7], this is equivalent to the approximate amenability of $l^1(G_{p_\lambda})$, hence to the amenability of G_{p_λ} for each $\lambda \in \Lambda$ ([4, Theorem 3.2]). Conversely, if G_{p_λ} is amenable, then $l^1(G_{p_\lambda})$ is amenable and so it is biflat. By [8, Proposition 2.7], $\mathcal{M}_{E(D_\lambda)}(l^1(G_{p_\lambda}))$ is biflat too. Now by [7, Theorem 3.3] we conclude that $\mathcal{M}_{E(D_\lambda)}(l^1(G_{p_\lambda}))$ is approximately biprojective. Therefore, by Proposition 2.1, $l^1(S)$ is approximately biprojective.

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(ii) The proof of this part is similar to that of part (i). □

Let G be a group with identity e, and let I be a non-empty set. Then the *Brandt inverse semigroup corresponding to* G *and* I, denoted $S = \mathcal{M}(G, I)$, is the collection of all $I \times I$ matrices $(g)_{ij}$ with $g \in G$ in the $(i, j)^{\text{th}}$ place and 0 (zero) elsewhere and the $I \times I$ zero matrix 0. Multiplication in S is given by the formula

$$(g)_{ij}(h)_{kl} = \begin{cases} (gh)_{il} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \qquad (g, h \in G,\, i, j, k, l \in I),$$

and ${(g)_{ij}}^* = (g^{-1})_{ji}$ and $0^* = 0$. The set of all idempotents is $E(S) = \{(e)_{ii} : i \in I\} \bigcup \{0\}$, and $G_{(e)_{ii}} \cong G$. It can be easily seen that each Brandt inverse semigroup is uniformly locally finite.

Corollary 2.7. *Let $S = \mathcal{M}(G, I)$ be the Brandt semigroup corresponding to G and I. Then $l^1(S)$ is (boundedly) approximately biprojective if and only if G is amenable.*

We remark that since $l^1(S)$ has the bounded approximation property, as we mentioned in the Introduction, the bounded approximate biprojectivity of $l^1(S)$ is equivalent to its biflatness. So the bounded case of Theorem 2.6 and Corollary 2.7 were obtained in [8].

3. Approximate biprojectivity of some algebras over topological Brandt semigroups

In this section, we study biprojectivity, biflatness and approximate biprojectivity of some Banach algebras related to topological Brandt semigroups. First we recall an analogue of measure algebra and group algebra for a topological semigroup. For a locally compact Hausdorff topological semigroup S, let $M(S)$ be the Banach space of all bounded complex-valued Radon measures on S with the total variation norm. With the convolution product $M(S)$ is a Banach algebra. We also set

$$M_a^l(S) = \{\mu \in M(S) \,|\, s \mapsto \delta_s * \mu \text{ is weakly continuous}\},$$

$$M_a^r(S) = \{\mu \in M(S) \,|\, s \mapsto \mu * \delta_s \text{ is weakly continuous}\},$$

$$M_a(S) = M_a^l(S) \cap M_a^r(S),$$

where δ_s denotes the Dirac measure at $s \in S$. It is well known that $M_a^l(S)$ and $M_a^r(S)$ are two sided L-ideals of $M(S)$ (see Lemma 3.5 and proof of Theorem 3.6 in [11]). It should be noted that in the case, where $S = G$ is a locally compact group, then $M_a(G) = M_a^l(G) = M_a^r(G) = L^1(G)$.

From now on, we take S to be $\mathcal{M}(G, I)$, a locally compact Hausdorff topological Brandt semigroup. For $A \subseteq G$ and $i, j \in I$ the set $\{(g)_{ij} : g \in A\}$ is denoted by $A_{i,j}$. In the case where $A = G$, the sets $G_{i,j}$ are open and homeomorphic, but not subsemigroups of S, unless $i = j$. Indeed, for each $i \in I$, $G_{i,i}$ is a maximal subgroup of S and also a locally compact topological group. Note that $G_{i,i}$'s are topologically isomorphic. Now fix $i_0 \in I$ and set

$$\tau_{(G,S)} = \{U \subseteq G : \; U_{i_0,i_0} \text{ is an open subset of } G_{i_0,i_0}\}.$$

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Then $\tau_{(G,S)}$ is a topology on G turning it to a locally compact group. Notice that all subsets $G_{i,j}$ are homeomorphic to $(G, \tau_{(G,S)})$ and maximal subgroups $G_{i,i}$ are isomorphic to $(G, \tau_{(G,S)})$ as locally compact groups.

For $i, j \in I$, $\mu \in M(S)$ and $\nu \in M(G)$, we define

$$\mu_{i,j}(A) = \mu(A_{i,j}) \quad \text{and} \quad \nu^{i,j}(B) = \nu(B^{i,j}),$$

where A is a Borel subset of G, B is a Borel subset of S and $B^{i,j} = \{g \in G : (g)_{i,j} \in B\}$. We need the following two propositions ([10, Propositions 3 and 5]).

Proposition 3.1. *Let $S = \mathcal{M}(G, I)$ be a locally compact Hausdorff topological Brandt semigroup and G be equipped with the topology $\tau_{(G,S)}$.*

(i) *For each $i, j \in I$ and $\mu \in M(G)$, $\mu^{i,j}$ is a well-defined measure in $M(S)$ such that $|\mu^{i,j}| = |\mu|^{i,j}$, $\|\mu^{i,j}\| = \|\mu\|$, and $\operatorname{supp}(\mu^{i,j}) = (\operatorname{supp}\mu)_{i,j}$.*

(ii) *If $\mu, \nu \in M(G)$ and $i, j, k, l \in I$, then $\mu^{i,j} * \nu^{k,l} = (\mu * \nu)^{i,l}$ for $j = k$, and $\mu^{i,j} * \nu^{k,l} = \mu(G)\nu(G)\delta_0$ for $j \neq k$. Also, $\delta_0 * \nu^{k,l} = \nu^{k,l} * \delta_0 = \nu(G)\delta_0$.*

(iii) *Let $\mu \in L^1(G)$ and $i, j \in I$. If for some $l_0 \in I$, $\bigcup_{i\in I} G_{i,l_0}(\bigcup_{i\in I} G_{l_0,i}$, respectively) is closed, then $\mu^{i,j} \in M_a^l(S)(\mu^{i,j} \in M_a^r(S)$, respectively).*

(iv) *For each $i, j \in I$ and $\mu \in M(S)$, $\mu_{i,j}$ is a well-defined measure in $M(G)$ and $|\mu_{i,j}| = |\mu|_{i,j}$.*

(v) *$\delta_0 \in M_a(S)$.*

(vi) *If $\mu \in M_a^l(S)(\mu \in M_a^r(S)$, respectively) and $i, j \in I$, then $\mu_{i,j}^{i,j} \in M_a^l(S)$ ($\mu_{i,j}^{i,j} \in M_a^r(S)$, respectively), and $\mu_{i,j} \in L^1(G)$.*

Proposition 3.2. *Let $S = \mathcal{M}(G, I)$ be a locally compact Hausdorff topological Brandt semigroup and G be equipped with the topology $\tau_{(G,S)}$. Then*

$$M(S) = \left\{ \sum_{i,j\in I} \mu_{i,j}^{i,j} + c_0\delta_0 : [\mu_{i,j}]_{i,j} \in \mathcal{M}_I(M(G)), c_0 \in \mathbb{C} \right\}$$

Moreover, if for some $l_0 \in I$, $\bigcup_{i\in I} G_{i,l_0}(\bigcup_{i\in I} G_{l_0,i}$, respectively) is closed, then $M_a^l(S)$ ($M_a^r(S)$, respectively) coincides with

$$\left\{ \sum_{i,j\in I} \mu_{i,j}^{i,j} + c_0\delta_0 : [\mu_{i,j}]_{i,j} \in \mathcal{M}_I(L^1(G)), c_0 \in \mathbb{C} \right\},$$

otherwise $M_a^l(S) = \mathbb{C}\delta_0$ ($M_a^r(S) = \mathbb{C}\delta_0$, respectively), and $M_a(S) = \mathbb{C}\delta_0$.

Using above propositions, in the following theorem, we characterize $M(S)$ as a direct sum of $\mathcal{M}_I(M(G))$ and $\mathbb{C}$. The idea is taken from [10, Proposition 4].

Theorem 3.3. *Let $\Theta : \mathcal{M}_I(M(G)) \oplus_{l^1} \mathbb{C} \to M(S)$ be defined by*

$$\Theta([\mu_{i,j}]_{i,j}, c) = \sum_{i,j\in I} \mu_{i,j}^{i,j} - \sum_{i,j\in I} \mu_{i,j}(G)\delta_0 + c\delta_0.$$

Then Θ is a Banach algebra isomorphism.

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Proof. Define the maps $\Phi : \mathcal{M}_I(M(G)) \to M(S)$ and $\Psi : M(S) \to \mathbb{C}$ by

$$\Phi([\mu_{i,j}]_{i,j}) = \sum_{i,j\in I} \mu_{i,j}^{i,j} - \sum_{i,j\in I} \mu_{i,j}(G)\delta_0 \quad \text{and} \quad \Psi(\mu) = \mu(S).$$

Clearly, Ψ is a continuous epimorphism. To see that Φ is a homomorphism, let $[\mu_{i,j}]_{i,j}, [\nu_{i,j}]_{i,j} \in \mathcal{M}_I(M(G))$. Then

$$\begin{aligned}
\Phi([\mu_{i,j}]_{i,j}) * \Phi([\nu_{i,j}]_{i,j}) &= \Big(\sum_{i,l\in I} \mu_{i,l}^{i,l} - \sum_{i,l\in I} \mu_{i,l}(G)\delta_0\Big)\Big(\sum_{k,j\in I} \nu_{k,j}^{k,j} - \sum_{k,j\in I} \nu_{k,j}(G)\delta_0\Big)\\
&= \sum_{i,l\in I}\sum_{k,j\in I} \mu_{i,l}^{i,l} * \nu_{k,j}^{k,j} - \Big(\sum_{i,l\in I} \mu_{i,l}(G)\Big)\Big(\sum_{k,j\in I} \nu_{k,j}(G)\Big)\delta_0\\
&- \Big(\sum_{i,l\in I} \mu_{i,l}(G)\Big)\Big(\sum_{k,j\in I} \nu_{k,j}(G)\Big)\delta_0 + \Big(\sum_{i,l\in I} \mu_{i,l}(G)\Big)\Big(\sum_{k,j\in I} \nu_{k,j}(G)\Big)\delta_0 * \delta_0\\
&= \sum_{i,l\in I}\sum_{k,j\in I} \mu_{i,l}^{i,l} * \nu_{k,j}^{k,j} - \Big(\sum_{i,l\in I} \mu_{i,l}(G)\Big)\Big(\sum_{k,j\in I} \nu_{k,j}(G)\Big)\delta_0\\
&= \sum_{i,j,k\in I} \mu_{i,k}^{i,k} * \nu_{k,j}^{k,j} + \sum_{i\neq k, i,j,k,l\in I} \mu_{i,l}^{i,l} * \nu_{k,j}^{k,j} - \sum_{i,j,k,l\in I} \mu_{i,l}(G)\nu_{k,j}(G)\delta_0\\
&= \sum_{i,j,k\in I} \mu_{i,k}^{i,k} * \nu_{k,j}^{k,j} + \sum_{k\neq l, i,j,k,l\in I} \mu_{i,l}(G)\nu_{k,j}(G)\delta_0 - \sum_{i,j,k,l\in I} \mu_{i,l}(G)\nu_{k,j}(G)\delta_0\\
&= \sum_{i,j,k\in I} \mu_{i,k}^{i,k} * \nu_{k,j}^{k,j} - \sum_{k=l, j,k,l\in I} \mu_{i,l}(G)\nu_{k,j}(G)\delta_0\\
&= \sum_{i,j\in I}\sum_{k\in I} \mu_{i,k}^{i,k} * \nu_{k,j}^{k,j} - \sum_{i,j\in I}\sum_{k\in I} \mu_{i,k}(G)\nu_{k,j}(G)\delta_0\\
&= \sum_{i,j\in I}\sum_{k\in I} (\mu_{i,k} * \nu_{k,j})^{i,j} - (\sum_{i,j\in I}(\sum_{k\in I} \mu_{i,k} * \nu_{k,j})(G))\delta_0\\
&= \Phi([\sum_{k\in I} \mu_{i,j} * \nu_{k,j}]_{i,j})\\
&= \Phi([\mu_{i,j}]_{i,j}[\nu_{i,j}]_{i,j}).
\end{aligned}$$

By noting that $S^{i,j} = G$ for any $i, j \in I$, we have

$$\Psi\Phi([\mu_{i,j}]_{i,j}) = \sum_{i,j\in I} \mu_{i,j}^{i,j}(S) - \sum_{i,j\in I} \mu_{i,j}(G) = \sum_{i,j\in I} \mu_{i,j}(G) - \sum_{i,j\in I} \mu_{i,j}(G) = 0,$$

which shows that $\mathrm{Im}(\Phi) \subseteq \ker(\Psi)$. For the converse inclusion, let $\mu \in \ker \Psi$. By Proposition 3.2, μ is of the form $\mu = \sum_{i,j\in I} \mu_{i,j}^{i,j} + c_0\delta_0$, where $[\mu_{i,j}]_{i,j} \in \mathcal{M}_I(M(G))$ and $c_0 \in \mathbb{C}$. Since

$$0 = \mu(S) = \sum_{i,j\in I} \mu_{i,j}^{i,j}(S) + c_0 = \sum_{i,j\in I} \mu_{i,j}(G) + c_0,$$

we have

$$\mu = \sum_{i,j\in I} \mu_{i,j}^{i,j} - \sum_{i,j\in I} \mu_{i,j}(G)\delta_0 = \Phi([\mu_{i,j}]_{i,j}) \in \mathrm{Im}(\Phi).$$

Therefore, $\mathrm{Im}(\Phi) = \ker(\Psi)$ and so $\mathrm{Im}(\Phi)$ is closed in $M(S)$. To see that Φ is injective, let $[\mu_{i,j}]_{i,j} \in \ker(\Phi)$. Then $\sum_{i,j\in I} \mu_{i,j}^{i,j} - \sum_{i,j\in I} \mu_{i,j}(G)\delta_0 = 0$. Thus, for any Borel set $E \subseteq G$ and any $i_0, j_0 \in I$,

$$0 = (\sum_{i,j\in I} \mu_{i,j}^{i,j} - \sum_{i,j\in I} \mu_{i,j}(G)\delta_0)(E^{i_0,j_0}) = \mu_{i_0,j_0}^{i_0,j_0}(E^{i_0,j_0}) = \mu_{i_0,j_0}(E). \tag{3.1}$$

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Therefore, $[\mu_{i,j}]_{i,j} = 0$. A direct calculation shows that

$$\mathrm{Im}(\Phi) \subseteq \{\mu \in M(S) : \mu * \delta_0 = \delta_0 * \mu = 0\}.$$

This together with $\delta_0^2 = \delta_0$ imply that Θ is a homomorphism. Indeed,

$$\begin{aligned}
\Theta([\mu_{i,j}]_{i,j}, c) * \Theta([\nu_{i,j}]_{i,j}, d) &= (\Phi([\mu_{i,j}]_{i,j}) + c\delta_0) * (\Phi([\nu_{i,j}]_{i,j}) + d\delta_0) \\
&= \Phi([\mu_{i,j}]_{i,j})(\Phi([\nu_{i,j}]_{i,j}) + cd\delta_0 \\
&= \Phi([\mu_{i,j}]_{i,j}[\nu_{i,j}]_{i,j}) + cd\delta_0 \\
&= \Theta([\mu_{i,j}]_{i,j}[\nu_{i,j}]_{i,j}, cd) \\
&= \Theta(([\mu_{i,j}]_{i,j}, c)([\nu_{i,j}]_{i,j}, d)).
\end{aligned}$$

Since $\delta_0 \in \mathrm{Im}(\Theta)\backslash\mathrm{Im}(\Phi)$, we have $\mathrm{Im}(\Phi) \subsetneq \mathrm{Im}(\Theta)$. Since $\dim(\frac{M(S)}{\mathrm{Im}(\Phi)}) = 1$, it follows that $\mathrm{Im}(\Theta) = M(S)$. Also, it is clear that Θ is injective. Therefore, Θ is an isomorphism of Banach algebras. □

Corollary 3.4. *Let $S = \mathcal{M}(G, I)$ be a locally compact Hausdorff topological Brandt semigroup and G be equipped with the topology $\tau_{(G,S)}$ in such a way that for some $l_0 \in I$, $\bigcup_{i\in I} G_{i,l_0}$($\bigcup_{i\in I} G_{l_0,i}$, respectively) is closed. Then the isomorphism $M(S) \cong \mathcal{M}_I(M(G)) \oplus_{l^1} \mathbb{C}$ induces the Banach algebra isomorphism*

$$M_a^l(S) \cong \mathcal{M}_I(L^1(G)) \oplus_{l^1} \mathbb{C} \quad (M_a^r(S) \cong \mathcal{M}_I(L^1(G)) \oplus_{l^1} \mathbb{C}).$$

Theorem 3.5. *Let $S = \mathcal{M}(G, I)$ be a locally compact Hausdorff topological Brandt semigroup and G be equipped with the topology $\tau_{(G,S)}$. The following are equivalent:*

(i) *$M(S)$ is boundedly approximately biprojective.*
(ii) *$M(S)$ is approximately biprojective.*
(iii) *G is discrete and amenable.*

Proof. (i) $\Longrightarrow$ (ii) is trivial.

(ii) $\Longrightarrow$ (iii): Since $M(S) \cong \mathcal{M}_I(M(G)) \oplus_{l^1} \mathbb{C}$ as Banach algebras, by Propositions 2.1 and 2.5, approximate biprojectivity of $M(S)$ implies the approximate biprojectivity of $M(G)$. This together with [7, Example 4.5] show that G is discrete and amenable.

(iii) $\Longrightarrow$ (i): If G is discrete and amenable, then $M(G)$ is amenable and so bounded approximately biprojective by [7, Corollary 3.10]. Now it follows from Propositions 2.5 and 2.1 and the isomorphism $M(S) \cong \mathcal{M}_I(M(G)) \oplus_{l^1} \mathbb{C}$ that $M(S)$ is boundedly approximately biprojective. □

The following corollary is the topological version of [8, Theorem 3.7], when S is a Brandt semigroup.

Corollary 3.6. *Let $S = \mathcal{M}(G, I)$ be a locally compact Hausdorff topological Brandt semigroup and G be equipped with the topology $\tau_{(G,S)}$. Then $M(S)$ is biflat if and only if G is discrete and amenable.*

Proof. Since $M(S)$ has the bounded approximation property, the corollary follows from Theorem 3.5, [1, Theorem 3.6(A)] and [7, Theorem 3.3]. □

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Applying [8, Proposition 2.7], one can easily observe that $M(S)$ is biprojective if and only if G is finite. The next corollary is a generalization of [3, Theorem 1.3], when S is a Brandt semigroup.

Corollary 3.7. *Let $S = \mathcal{M}(G, I)$ be a locally compact Hausdorff topological Brandt semigroup and G be equipped with the topology $\tau_{(G,S)}$. Then $M(S)$ is amenable if and only if I is finite and G is discrete and amenable.*

Proof. The isomorphism $M(S) \cong \mathcal{M}_I(M(G)) \oplus_{l^1} \mathbb{C}$ shows that $M(S)$ has a bounded approximate identity if and only if I is finite. Now Theorem 3.5 and [7, Corollary 3.10] complete the proof. $\square$

For a locally compact group G the condition $M(G) = l^1(G)$ implies the discreteness of G, but this phenomenon does not hold for a locally compact semigroup S. Thus the discreteness of G in Theorem 3.5 does not necessarily induce the discreteness of S. Nonetheless, we have the following corollary by applying [10, Corollary 2].

Corollary 3.8. *Let $S = \mathcal{M}(G, I)$ be a locally compact Hausdorff topological Brandt semigroup and G be equipped with the topology $\tau_{(G,S)}$. Let $M(S)$ be approximately biprojective. Then S is a fundamental semigroup and $M(S) = M_a(S) = l^1(S)$.*

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