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MODULI OF J-HOLOMORPHIC CURVES WITH LAGRANGIAN BOUNDARY CONDITIONS AND OPEN GROMOV-WITTEN INVARIANTS FOR AN S^1-EQUIVARIANT PAIR

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Abstract. Let (X, ω) be a symplectic manifold, J be an ω-tame almost complex structure, and L be a Lagrangian submanifold. The stable compactification of the moduli space of parametrized J-holomorphic curves in X with boundary in L (with prescribed topological data) is compact and Hausdorff in Gromov's C^∞-topology. We construct a Kuranishi structure with corners in the sense of Fukaya and Ono. This Kuranishi structure is orientable if L is spin. In the special case where the expected dimension of the moduli space is zero, and there is an S^1-action on the pair (X, L) which preserves J and has no fixed points on L, we define the Euler number for this S^1-equivariant pair and the prescribed topological data. We conjecture that this rational number is the one computed by localization techniques using the given S^1-action.

1. Introduction

1.1. **Background.** String theorists have been making predictions on enumerative invariants using dualities. One of the most famous examples is the astonishing predictions of the number of rational curves in a quintic threefold in [8]. To understand these predictions, mathematicians first developed Gromov-Witten theory to give the numerical invariants a rigorous mathematical definition so that these predictions could be formulated as mathematical statements, and then tried to prove these statements. The predictions in [8] are proven in [11, 25].

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Later, string theorists have produced enumerative predictions about holomorphic curves with Lagrangian boundary conditions by studying dualities involving open strings [4, 5, 30, 32, 33]. Moreover, assuming the existence of a virtual fundamental cycle and the validity of virtual localization, mathematicians have carried out computations which coincide with these predictions [14, 24]. In certain cases, these numbers can be reproduced by considering relative morphisms [26]. It is desirable to give a rigorous mathematical definition of these enumerative invariants, so that we may formulate physicists' predictions as mathematical theorems, and then try to prove these theorems.

Gromov-Witten invariants count J-holomorphic maps from a Riemann surface to a fixed symplectic manifold (X, ω) together with an ω-tame almost complex structure J. These numbers can be viewed as intersection numbers on the moduli space of such maps. We want the moduli space to be compact without boundary and oriented so that there exists a fundamental cycle which allows us to do intersection theory. The moduli space of J-holomorphic maps can be compactified by adding "stable maps", whose domain is a Riemann surface which might have nodal singularities. The stable compactification is compact and Hausdorff in the C^∞ topology defined by Gromov [15]. The moduli space is essentially almost complex, so it has a natural orientation. In general, the moduli space is not of the expected dimension and has bad singularities, but there exists a "virtual fundamental cycle" which plays the role of fundamental cycles [7, 9, 27, 28, 37]. These are now well-established in Gromov-Witten theory.

The "open Gromov-Witten invariants" that we want to establish shall count J-holomorphic maps from a bordered Riemann surface to a symplectic manifold X as above such that the image of the boundary lies in a Lagrangian submanifold L of X. To compactify the moduli space of such maps, Sheldon Katz and the author [24] introduced stable maps in this context. The stable compactification is compact and Hausdorff in the C^∞ topology, as in the ordinary Gromov-Witten theory. However, orientation is a nontrivial issue in the open Gromov-Witten theory. Moreover, the boundary is of real codimension one, so the compactified moduli space does not "close up" as in the ordinary Gromov-Witten theory, and the best we can expect is a fundamental "chain".

A fundamental "chain" is not satisfactory for intersection theory. For example, the Euler characteristic of a compact oriented manifold without boundary can be defined as the number of zeros of a generic vector field, counted with signs determined by the orientation. This number is independent of the choice of the vector field, and thus well-defined. For a compact oriented manifold with boundary, one can still count the number of zeros of a generic vector field with signs determined by the orientation, but the number will depend on the choice of the vector field. Therefore, we need to specify extra boundary conditions to get a well-defined number.

1.2. **Main results.** Let (X, ω) be a symplectic manifold of dimension $2N$, and L be a Lagrangian submanifold. To compactify the moduli space of parametrized J-holomorphic curves in X with boundary in L, Gromov introduced cusp curves with boundary [15], which are called here prestable maps. A *prestable map* to (X, L) is a continuous map $f : (\Sigma, \partial\Sigma) \to (X, L)$ such that $f \circ \tau : (\hat{\Sigma}, \partial\hat{\Sigma}) \to (X, L)$ is J-holomorphic, where Σ is a prestable (i.e., smooth or nodal) bordered Riemann surface, and $\tau : \hat{\Sigma} \to \Sigma$ is the normalization map [24, Definition 3.6.2].

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A smooth bordered Riemann surface Σ is of type (g,h) if it is topologically a sphere with g handles and h holes. The boundary of Σ consists of h disjoint circles $B^1,\ldots,B^h$. We say Σ has $(n,\vec{m})$ marked points if there are n distinct marked points in its interior and m^i distinct marked points on B^i, where $\vec{m}=(m^1,\ldots,m^h)$, $m^i\geq 0$. By allowing nodal singularities, we have the notion of a prestable Riemann surface of type (g,h) with $(n,\vec{m})$ marked points and an ordering $B^1,\ldots,B^h$ of the boundary components. An isomorphism between two such prestable bordered Riemann surfaces is an isomorphism of prestable bordered Riemann surfaces which preserves the marked points and ordering of boundary components. An isomorphism between two prestable maps $f:\Sigma\to X$ and $f':\Sigma'\to X$ is an isomorphism $\phi:\Sigma\to\Sigma'$ in the above sense such that $f=f'\circ\phi$. A prestable map is *stable* if its automorphism group is finite. This is the analogue of Kontsevich's stable maps [23] in the ordinary Gromov-Witten theory.

For $\beta\in H^2(X,L;\mathbb{Z})$, $\vec{\gamma}=(\gamma^1,\ldots,\gamma^h)\in H^1(L;\mathbb{Z})^{\oplus h}$, and $\mu\in\mathbb{Z}$, define

$$\overline{M}_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)$$

to be the moduli space of isomorphism classes of stable maps $f:(\Sigma,\partial\Sigma)\to(X,L)$, where Σ is a prestable bordered Riemann surface of type (g,h) with $(n,\vec{m})$ marked points and an ordering $B^1,\ldots,B^h$ of the boundary components, $f_*[\Sigma]=\beta$, $f_*[B^i]=\gamma^i$, $i=1,\ldots,h$, and $\mu(f^*T_X,(f|_{\partial\Sigma})^*T_L)=\mu$. Here $\mu(f^*T_X,((f|_{\partial\Sigma})^*T_L)$ is the Maslov index defined in [24, Definition 3.3.7]. We have the following result, which is possibly part of the literature.

Theorem 1.1. *$\overline{M}_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)$ is compact and Hausdorff in the C^∞ topology.*

Here the C^∞ topology is the one defined by Gromov's weak convergence [15]. The stability condition is necessary for Hausdorffness. The compactness follows from [15, 47], which will be explained in Section 5.3.

The boundary of the moduli space corresponds to degeneration of the domain or blowup of the map. An interior node corresponds to a (real) codimension 2 stratum, while a boundary node corresponds to a codimension 1 stratum. Blowup of the map at an interior point leads to the well-known phenomenon of bubbling off of spheres which is codimension 2, while blowup at a boundary point leads to bubbling off of discs which is codimension 1. The intersection of two or more codimension 1 strata forms a corner. The next result is shown in Section 6.

Theorem 1.2. *$\overline{M}_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)$ has a Kuranishi structure with corners of (real) virtual dimension $\mu+(N-3)(2-2g-h)+2n+m^1+\cdots+m^h$, where $2N$ is the (real) dimension of X. The Kuranishi structure is orientable if L is spin or if $h=1$ and L is relatively spin (i.e., L is orientable and $w_2(T_L)=\alpha|_L$ for some $\alpha\in H^2(X,\mathbb{Z}_2)$).*

The case for the disc with only boundary marked points ($g=n=0$, $h=1$) is proven in [10]. Let us describe briefly what a Kuranishi structure with corners is, and refer to Section 6.1 for the complete definition. A chart of a Kuranishi structure with corners is a 5-uple (V,E,Γ,ψ,s), where V is a smooth manifold (possibly with corners), Γ is a finite group acting on V, E is a Γ-equivariant

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vector bundle over V, $s : V \to E$ is a Γ-equivariant section, and ψ maps $s^{-1}(0)/\Gamma$ homeomorphically to an open set of the moduli. The dimension of V, rank of E, and the finite group Γ might vary with charts, but $d = \dim V - \mathrm{rank} E$ is fixed and is the virtual dimension of the Kuranishi structure with corners. Now $\det(TV) \otimes (\det E)^{-1}$ can be glued to an orbibundle, the orientation bundle, and the Kuranishi structure with corners is orientable if its orientation bundle is a trivial real line bundle.

If s intersects the zero section transversally, $s^{-1}(0)$ is a manifold (possibly with corners) of dimension d. In general, $s^{-1}(0)$ might have dimension larger than d and bad singularities due to the nontransversality of s. The virtual fundamental chain can be constructed by perturbing s to a transversal section. Locally it is a singular chain with rational coefficients in V/Γ which is a rational combination of the images of d-dimensional submanifolds of V.

A virtual fundamental chain is not satisfactory for intersection theory. For example, when X is a Calabi-Yau threefold and L is a special Lagrangian submanifold, $\overline{M}_{(g,h),(0,\vec{0})}(X, L \mid \beta, \vec{\gamma}, \mu)$ is empty for $\mu \neq 0$, and the expected dimension of $\overline{M}_{(g,h),(0,\vec{0})}(X, L \mid \beta, \vec{\gamma}, 0)$ is zero for any g, h, β, $\vec{\gamma}$. The virtual fundamental chain is a zero chain with rational coefficients, and we would like to define the invariant $\chi_{(g,h)}(X, L \mid \beta, \vec{\gamma}, \mu) \in \mathbb{Q}$ to be the degree of this zero chain. However, this number depends on the perturbation, so we need to impose extra boundary conditions to obtain a well-defined number. Next assume that

(1) There is an S^1-action $\varrho : S^1 \times X \to X$ which preserves J and L.
(2) The restriction of ϱ to L is fixed point free.
(3) The virtual dimension of $\overline{M}_{(g,h),(0,\vec{0})}(X, L \mid \beta, \vec{\gamma}, \mu)$ is zero.

Note that (1) implies that S^1 acts on the moduli spaces $\overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$. We will only consider the case $(n, \vec{m}) = (0, \vec{0})$ (no marked points). Also (2) implies that the S^1-action on the moduli space has no fixed point in the union of corners. Under assumptions (1),(2),(3) one can, using the S^1-action, impose boundary conditions to get a well-defined rational number

$$\chi_{(g,h)}(X, L, \varrho \mid \beta, \vec{\gamma}, \mu).$$

The construction can be illustrated by a toy model. We first consider a toy model of a space with an oriented Kuranishi structure (without corners) of virtual dimension zero. Let E be a rank r oriented vector bundle over a compact, connected, oriented r dimensional manifold V. Let $s : V \to E$ be a smooth section, and let $M = s^{-1}(0)$. Then $(V, E, \{1\}, \psi, s)$ defines an orientable Kuranish structure of virtual dimension 0 on M, where $\psi : s^{-1}(0) \to M$ is the identity map. Let $s' : V \to E$ be a smooth section transversal to the zero section. The number of zeros of s', counted with signs determined by orientation, is independent of choice of s'. Replacing V by a compact manifold with boundary, we get a toy model of a space with an oriented Kuranishi structure with corners of virtual dimension zero. The number of zeros of a section $s' : V \to E$ transversal to the zero section (counted with signs) depends on the choice of s'. However, suppose that there is an S^1 action on V which preserves ∂V and acts freely on ∂V, and suppose that E and s are S^1-equivariant, we may require s' be transversal to the zero section and satisfy an extra boundary condition: $s'|_{\partial V}$ is never zero and is S^1-equivariant. The number of zeros of such a section s' (counted with signs) is independent of choice of s'. This number is an invariant of the S^1-equivariant vector bundle $E \to V$ but *not* an invariant of $E \to V$. If $E \to V$ admits more than one S^1 action, different S^1 actions may define different numbers.

Similarly, $\chi_{(g,h)}(X, L, \varrho \mid \beta, \vec{\gamma}, \mu)$ is an invariant of the equivariant pair (X, L, ϱ), but *not* an invariant of the pair (X, L). I conjecture that these rational numbers are the ones computed by localization techniques using the S^1 action ϱ [14, 24]. The computations in [14, 24, 26] coincide with physicists' predictions.

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2. Surfaces with Analytic or Dianalytic Structures

In this section, we review some definitions and facts on surfaces with analytic or dianalytic structures, following [3, Chapter 1] closely. This section is an expansion of Section 3.1 and 3.2 of [24].

The *marked bordered Riemann surfaces* defined in Section 2.2.5 are directly related to open Gromov-Witten theory.

2.1. Analyticity and dianalyticity.

Definition 2.1. *A map $f : A \to \mathbb{C}$ is analytic on A if $\frac{\partial f}{\partial \bar{z}} = 0$, antianalytic on A if $\frac{\partial f}{\partial z} = 0$ and dianalytic on A if its restriction to each component of A is either analytic of antianalytic.*

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Definition 2.2. *Let A and B be nonempty subsets of $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \mathrm{Im} z \geq 0\}$. A continuous function $f : A \to B$ is* analytic (*resp.* antianalytic) *on A if it extends to an* analytic (*resp.* antianalytic) *function $f_{\mathbb{C}} : U \to \mathbb{C}$, where U is an open neighborhood of A in $\mathbb{C}$. f is said to be* dianalytic *on A if its restriction to each component of A is either analytic or antianalytic.*

Theorem 2.3 (Schwarz reflection principle)**.** *Let A and B be nonempty subsets of $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \mathrm{Im} z \geq 0\}$. A continuous function $f : A \to B$ is dianalytic* (*resp. analytic*) *if it is dianalytic* (*resp. analytic*) *on the interior of A and satisfies $f(A \cap \mathbb{R}) \subset \mathbb{R}$.*

Definition 2.4. *A* surface *is a Hausdorff, connected, topological space Σ together with a family $\mathcal{A} = \{(U_i, \phi_i) \mid i \in I\}$ such that $\{U_i \mid i \in I\}$ is an open covering of Σ and each map $\phi_i : U_i \to A_i$ is a homeomorphism onto an open subset A_i of $\mathbb{C}^+$. $\mathcal{A}$ is called a* topological atlas *on Σ, and each pair (U_i, ϕ_i) is called a* chart *of $\mathcal{A}$. The boundary of Σ is the set*

$$\partial\Sigma = \{x \in \Sigma \mid \exists\, i \in I \text{ s.t. } x \in U_i, \phi_i(x) \in \mathbb{R}\}$$

and $\phi_{ij} \equiv \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ are surjective homeomorphisms, called the transition functions *of $\mathcal{A}$. $\mathcal{A}$ is called a* dianalytic (*resp.* analytic) atlas *if all its transition functions are dianalytic* (*resp. analytic*).

2.2. **Various categories of surfaces.**

2.2.1. *Riemann surfaces.*

Definition 2.5. *A* Riemann surface *is a surface equipped with the analytic structure induced by an analytic atlas.*

A Riemann surface is canonically oriented by its analytic structure.

2.2.2. *Symmetric Riemann surfaces.*

Definition 2.6. *A* symmetric Riemann surface *is a Riemann surface Σ together with an antiholomorphic involution $\sigma : \Sigma \to \Sigma$, called the* symmetry *of Σ.*

Definition 2.7. *A* morphism *between symmetric Riemann surfaces (Σ, σ) and (Σ', σ') is an analytic map $f : \Sigma \to \Sigma'$ such that $f \circ \sigma = \sigma' \circ f$.*

Definition 2.8. *A* symmetric Riemann surface with (n, m) marked points *is a symmetric Riemann surface (Σ, σ) together with $2n + m$ distinct points $p_1, \ldots, p_{2n+m}$ in Σ such that $\sigma(p_i) = p_{n+i}$ for $i = 1, \ldots, n$ and $\sigma(p_i) = p_i$ for $i = 2n + 1, \ldots, 2n + m$.*

2.2.3. *Klein surfaces.*

Definition 2.9. *A* Klein surface *is a surface equipped with the dianalytic structure induced by a dianalytic atlas.*

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A Riemann surface can be viewed as a Klein surface. A Klein surface can be equipped with an analytic structure compatible with the dianalytic structure if and only if it is orientable. In particular, an orientable Klein surface without boundary admits a compatible structure of a Riemann surface.

Definition 2.10. *A* morphism *between Klein surfaces* Σ *and* Σ' *is a continuous map* $f : (\Sigma, \partial\Sigma) \to (\Sigma', \partial\Sigma')$ *such that for any* $x \in \Sigma$ *there exist analytic charts* (U, ϕ) *and* (V, ψ) *about* x *and* $f(x)$ *respectively, and an analytic function* $F : \phi(U) \to \mathbb{C}$ *such that the following diagram commutes:*

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[d, "\phi"] & & V \arrow[d, "\psi"] \\ \phi(U) \arrow[r, "F"] & \mathbb{C} \arrow[r, "\Phi"] & \mathbb{C}^+ \end{tikzcd}$$

where $\Phi(x + iy) = x + i|y|$ *is the* folding map.

Given a Klein surface Σ, there are three ways to construct an unramified double cover of Σ. We refer to [3, 1.6] for the precise definition of an unramified double cover and detailed constructions. The *complex double* $\Sigma_{\mathbb{C}}$ is an orientable Klein surface without boundary. The *orienting double* $\Sigma_{\mathbf{O}}$ is an orientable Klein surface. It is disconnected if and only if Σ is orientable, and it has nonempty boundary if and only of Σ has nonempty boundary. The *Schottkey double* $\Sigma_{\mathbf{S}}$ is a Klein surface without boundary. It is disconnected if and only if the boundary of Σ is empty, and it is nonorientable if and only if Σ is nonorientable.

If Σ is orientable, then $\Sigma_{\mathbb{C}} = \Sigma_{\mathbf{S}}$, and $\Sigma_{\mathbf{O}}$ is disconnected (the trivial double cover). If $\partial\Sigma = \phi$, then $\Sigma_{\mathbb{C}} = \Sigma_{\mathbf{O}}$, and $\Sigma_{\mathbf{S}}$ is disconnected. In particular, if Σ comes from a Riemann surface, then these three covers are the trivial disconnected double cover.

Example 2.11. *Let* Σ *be a Möbius strip. Then* $\Sigma_{\mathbb{C}}$ *is a torus,* $\Sigma_{\mathbf{S}}$ *is a Klein bottle, and* $\Sigma_{\mathbf{O}}$ *is an annulus.*

2.2.4. *Bordered Riemann surfaces.*

Definition 2.12. *A* bordered Riemann surface *is a compact surface with nonempty boundary equipped with the analytic structure induced by an analytic atlas.*

Remark 2.13. *A bordered Riemann surface is canonically oriented by the analytic (complex) structure. In the rest of this paper, the boundary circles* B^i *of a bordered Riemann surface* Σ *with boundary* $\partial\Sigma = B^1 \cup \ldots \cup B^h$ *will always be endowed with the orientation induced by the complex structure, which is a choice of tangent vector to* B^i *such that the basis (the tangent vector of* B^i*, inner normal) for the real tangent space is consistent with the orientation of* Σ *induced by the complex structure.*

Definition 2.14. *A* morphism *between bordered Riemann surfaces* Σ *and* Σ' *is a continuous map* $f : (\Sigma, \partial\Sigma) \to (\Sigma', \partial\Sigma')$ *such that for any* $x \in \Sigma$ *there exist analytic charts* (U, ϕ) *and* (V, ψ) *about* x *and* $f(x)$ *respectively, and an analytic function* $F : \phi(U) \to \mathbb{C}$ *such that the following diagram*

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commutes:

$$
\begin{array}{ccc}
U & \xrightarrow{f} & V \\
\Big\downarrow \phi & & \Big\downarrow \psi \\
\phi(U) & \xrightarrow{F} & \mathbb{C}
\end{array}
$$

A bordered Riemann surface is topologically a sphere with $g \geq 0$ handles and with $h > 0$ discs removed. Such a bordered Riemann surface is said to be of type (g, h).

A bordered Riemann surface can be viewed as a Klein surface. Its complex double and Schottkey double coincide since it is orientable.

2.2.5. *Marked bordered Riemann surfaces.* The following refinement of an earlier definition is suggested to the author by Ezra Getzler.

Definition 2.15. *Let h be a positive integer, g, n be nonnegative integers, and $\vec{m} = (m^1, \ldots, m^h)$ be an h-uple of nonnegative integers. A* marked bordered Riemann surface of type (g, h) with $(n, \vec{m})$ marked points *is an $(h+3)$-uple*

$$(\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h)$$

whose components are described as follows.

- Σ *is a bordered Riemann surface of type (g, h).*
- $\mathbf{B} = (B^1, \ldots, B^h)$, *where $B^1, \ldots, B^h$ are connected components of $\partial\Sigma$, oriented as in Remark 2.13.*
- $\mathbf{p} = (p_1, \ldots, p_n)$ *is an n-uple of distinct points in Σ°.*
- $\mathbf{q}^i = (q^i_1, \ldots, q^i_{m^i})$ *is an m^i-uple of distinct points on the circle B^i.*

Let $\vec{0} = (0, \ldots, 0)$. Note that a marked bordered Riemann surface of type (g, h) with $(n, \vec{0})$ marked points is a bordered Riemann surface together with an ordering of the h boundary components.

Definition 2.16. *A* morphism *between marked bordered Riemann surfaces of type (g, h) with $(n, \vec{m})$ marked points*

$$(\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h) \to (\Sigma', \mathbf{B}'; \mathbf{p}'; (\mathbf{q}')^1, \ldots, (\mathbf{q}')^h)$$

is an isomorphism of bordered Riemann surface $f : \Sigma \to \Sigma'$ such that $f(B^i) = (B')^i$ for $i = 1, \ldots, h$, $f(p_j) = p'_j$ for $j = 1, \ldots, n$, and $f(q^i_k) = (q')^i_k$ for $k = 1, \ldots, m^i$.

Remark 2.17. *The category of marked bordered Riemann surfaces of type (g, h) with $(n, \vec{m})$ marked points is a groupoid since every morphism in Definition 2.16 is an isomorphism.*

2.3. **Topological types of compact symmetric Riemann surfaces.** A compact symmetric Riemann surface is topologically a compact orientable surface without boundary Σ together with an orientation reversing involution σ, which is classified by the following three invariants:

(1) The genus $\tilde{g}$ of Σ.
(2) The number $h = h(\sigma)$ of connected components of Σ^σ, the fixed locus of σ.
(3) The index of orientability, $k = k(\sigma) :=$ 2−the number of connected components of $\Sigma \backslash \Sigma^\sigma$.

These invariants satisfy:

(1) $0 \leq h \leq \tilde{g}+1$.
(2) For $k=0$, $h>0$ and $h \equiv \tilde{g}+1 \pmod 2$.
(3) For $k=1$, $0 \leq h \leq \tilde{g}$.

The above classification was realized already by Felix Klein (see e.g. [22,36,44]). It is probably better understood in terms of the quotient $Q(\Sigma)=\Sigma/\langle\sigma\rangle$, where $\langle\sigma\rangle=\{id,\sigma\}$ is the group generated by σ. The quotient $Q(\Sigma)$ is orientable if $k=0$ and nonorientable if $k=1$, hence the name "index of orientability". Furthermore, h is the number of connected components of the boundary of $Q(\Sigma)$. If $Q(\Sigma)$ is orientable, then it is topologically a sphere with $g \geq 0$ handles and with $h>0$ discs removed, and the invariants of (Σ,σ) are $(\tilde{g},h,k)=(2g+h-1,h,0)$. If $Q(\Sigma)$ is nonorientable, then it is topologically a sphere with $g>0$ crosscaps and with $h \geq 0$ discs removed, and the invariants of Σ are $(\tilde{g},h,k)=(g+h-1,h,1)$.

From the above classification we see that symmetric Riemann surfaces of a given genus $\tilde{g}$ constitute $[\frac{3\tilde{g}+4}{2}]$ topological types.

3. Deformation theory of bordered Riemann rurfaces

In this section, we study deformation theory of bordered Riemann surfaces. We refer to [24, Section 3] for some preliminaries such as doubling constructions and the Riemann-Roch theorem for bordered Riemann surfaces.

3.1. Deformation theory of smooth bordered Riemann surfaces.

Let Σ be a bordered Riemann surface, $(\Sigma_{\mathbb{C}},\sigma)$ be its complex double (see e.g. [24, Section 3.3.1] for the definition). Analytically, $(\Sigma_{\mathbb{C}},\sigma)$ is a compact symmetric Riemann surface. Algebraically, it is a smooth complex algebraic curve X which is the complexification of some smooth real algebraic curve X_0, i.e., $X=X_0\times_{\mathbb{R}}\mathbb{C}$ (see [16, Chapter II, Exercise 4.7]). Alternatively, (X,S) is a complex algebraic curve with a real structure (see [39, I.1]), where S is a *semi-linear* automorphism in the sense of [16, Chapter II, Exercise 4.7] which induces the antiholomorphic involution σ on $\Sigma_{\mathbb{C}}$.

3.1.1. *Algebraic approach.*

First order deformation of the complex algebraic curve X is canonically identified with the complex vector space $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X,\mathcal{O}_X)$, where Ω_X is the sheaf of Kähler differentials on X. The obstruction lies in $\mathrm{Ext}^2_{\mathcal{O}_X}(\Omega_X,\mathcal{O}_X)=0$. Similarly, the first order deformation of the real algebraic curve X_0 is identified with the real vector space $\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0},\mathcal{O}_{X_0})$, and the obstruction vanishes. We have

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X,\mathcal{O}_X)\cong \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0},\mathcal{O}_{X_0})\otimes_{\mathbb{R}}\mathbb{C}$$

since $X=X_0\times_{\mathbb{R}}\mathbb{C}$. The semi-linear automorphism S induces a complex conjugation

$$S:\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X,\mathcal{O}_X)\to \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X,\mathcal{O}_X).$$

The fixed locus $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X,\mathcal{O}_X)^S$ gives the first order deformation of (X,S) as a complex algebraic curve with a real structure, and is naturally isomorphic to $\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0},\mathcal{O}_{X_0})$.

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More explicitly, X can be covered by complex affine curves which is a complete intersection of hypersurfaces defined by polynomials with real coefficients. Deformation of X are given by varying the coefficients (in $\mathbb{C}$). Deformation of (X,S) is given by varying the coefficients in $\mathbb{R}$. The above polynomials with real coefficients also define the real algebraic curve X_0, and varying the coefficients in $\mathbb{R}$ gives the deformation of X_0. The complex conjugation of coefficients corresponds to the above complex conjugation S on $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X,\mathcal{O}_X)$.

X is a smooth algebraic variety, thus $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X,\mathcal{O}_X)$ is isomorphic to the sheaf cohomology group $H^1(X,\Theta_X)$, where Θ_X is the tangent sheaf of X, and

$$\mathrm{Ext}^2_{\mathcal{O}_X}(\Omega_X,\mathcal{O}_X)\cong H^2(X,\Theta_X)=0.$$

Similarly, we have

$$\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0},\mathcal{O}_{X_0})\cong H^1(X,\Theta_{X_0}),\quad \mathrm{Ext}^2_{\mathcal{O}_{X_0}}(\Omega_{X_0},\mathcal{O}_{X_0})\cong H^2(X,\Theta_{X_0})=0.$$

We now return to the original bordered Riemann surface Σ.

Definition 3.1. *Let $\mathcal{O}_\Sigma$ be the sheaf of local holomorphic functions on Σ with real boundary values. Let Ω_Σ be a sheaf of $\mathcal{O}_\Sigma$-modules, together with an $\mathbb{R}$ derivation $d:\mathcal{O}_\Sigma\to\Omega_\Sigma$, which satisfy the following universal property: for any sheaf of $\mathcal{O}_\Sigma$-modules $\mathcal{F}$, and for any $\mathbb{R}$ derivation $d':\mathcal{O}_\Sigma\to\mathcal{F}$, there exists a unique $\mathcal{O}_\Sigma$-module homomorphism $f:\Omega_\Sigma\to\mathcal{F}$ such that $d'=f\circ d$. We call Ω_Σ the* sheaf of Kähler differentials on Σ.

Let $\Theta_\Sigma=\mathcal{H}om_{\mathcal{O}_\Sigma}(\Omega_\Sigma,\mathcal{O}_\Sigma)=\Omega_\Sigma^\vee$, be the dual of Ω_Σ in the category of sheaves of $\mathcal{O}_\Sigma$-modules.

Note that Ω_Σ and Θ_Σ are locally free sheaves of $\mathcal{O}_\Sigma$-modules of rank 1. Analytically, Ω_Σ is the sheaf of local holomorphic 1-forms on Σ whose restriction to $\partial\Sigma$ are real 1-forms, and Θ_Σ is the sheaf of holomorphic vector fields with boundary values in $T_{\partial\Sigma}$. There are natural isomorphisms

$$\mathrm{Ext}^i_{\mathcal{O}_\Sigma}(\Omega^1_\Sigma,\mathcal{O}_\Sigma)\cong\mathrm{Ext}^i_{\mathcal{O}_{X_0}}(\Omega^1_{X_0},\mathcal{O}_{X_0}),\quad H^i(\Sigma,\Theta_\Sigma)\cong H^i(X_0,\Theta_{X_0})$$

for $i\geq 0$. Since the first order deformations of the bordered Riemann surface Σ, of the symmetric Riemann surface $(\Sigma_{\mathbb{C}},\sigma)$, and of the real algebraic curve X_0 are identified, the first order deformation of Σ is canonically identified with

$$\mathrm{Ext}^1_{\mathcal{O}_\Sigma}(\Omega_\Sigma,\mathcal{O}_\Sigma)\cong H^1(\Sigma,\Theta_\Sigma),$$

and the obstruction lies in

$$\mathrm{Ext}^2_{\mathcal{O}_\Sigma}(\Omega_\Sigma,\mathcal{O}_\Sigma)\cong H^2(\Sigma,\Theta_\Sigma)=0.$$

3.1.2. *Analytic approach.* We first give the definitions of a *differentiable family of compact symmetric Riemann surfaces* and a *differentiable family of bordered Riemann surfaces*, which are modifications of [23, Definition 4.1].

Definition 3.2. *Suppose given a compact symmetric Riemann surface (M_t,σ_t) for each point t of a domain B of $\mathbb{R}^m$. $\{(M_t,\sigma_t)|t\in B\}$ is called a* differentiable family of symmetric Riemann surfaces *if* *there are a differentiable manifold $\mathcal{M}$, a surjective C^∞ map $\pi : \mathcal{M} \to B$ and a C^∞ map $\sigma : \mathcal{M} \to \mathcal{M}$ such that*

(1) *The rank of the Jacobian matrix of π is equal to m at every point of $\mathcal{M}$.*
(2) *For each $t \in B$, $\pi^{-1}(t)$ is a compact connected subset of $\mathcal{M}$.*
(3) *$\pi^{-1}(t) = M_t$.*
(4) *There are a locally finite open covering $\{\mathcal{U}_i \mid i \in I\}$ of $\mathcal{M}$ and C^∞ functions $z_i : \mathcal{U}_i \to \mathbb{C}$ such that*
$$\{(\mathcal{U}_i \cap \pi^{-1}(t), z_i|_{\mathcal{U}_i \cap \pi^{-1}(t)}) \mid i \in I, \mathcal{U}_i \cap \pi^{-1}(t) \neq \phi\}$$
is an analytic atlas for M_t.
(5) *$\pi \circ \sigma = \pi$, and $\sigma|_{\pi^{-1}(t)} = \sigma_t : M_t \to M_t$ is an antiholomorphic involution.*

Definition 3.3. *Suppose given a bordered Riemann surface M_t for each point t of a domain B of $\mathbb{R}^m$. $\{M_t \mid t \in B\}$ is called a* differentiable family of bordered Riemann surfaces *if there are a differentiable manifold with boundary $\mathcal{M}$ and a surjective C^∞ map $\pi : \mathcal{M} \to B$ such that*

(1) *The rank of the Jacobian matrix of π is equal to m at every point of $\mathcal{M}$.*
(2) *For each $t \in B$, $\pi^{-1}(t)$ is a compact connected subset of $\mathcal{M}$.*
(3) *$\pi^{-1}(t) = M_t$.*
(4) *There are a locally finite open covering $\{\mathcal{U}_i \mid i \in I\}$ of $\mathcal{M}$ and C^∞ functions $z_i : \mathcal{U}_i \to \mathbb{C}_+$ such that*
$$\{(\mathcal{U}_i \cap \pi^{-1}(t), z_i|_{\mathcal{U}_i \cap \pi^{-1}(t)}) \mid i \in I, \mathcal{U}_i \cap \pi^{-1}(t) \neq \phi\}$$
is an analytic atlas for M_t.

The complex double $\Sigma_{\mathbb{C}}$ of a bordered Riemann surface Σ is a complex manifold of dimension 1. Infinitesimal deformation of $\Sigma_{\mathbb{C}}$ can be identified with
$$H^1(\Sigma_{\mathbb{C}}, T_{\Sigma_{\mathbb{C}}}) = H^1(X, \Theta_X),$$
and the obstruction lies in
$$H^2(\Sigma_{\mathbb{C}}, T_{\Sigma_{\mathbb{C}}}) = H^2(X, \Theta_X) = 0.$$
(See [23].) The differential $d\sigma$ of σ is an antiholomorphic involution on the holomorphic line bundle $T_{\Sigma_{\mathbb{C}}} \to \Sigma_{\mathbb{C}}$ which covers $\sigma : \Sigma_{\mathbb{C}} \to \Sigma_{\mathbb{C}}$. $d\sigma$ induces a complex conjugation
$$\tilde{\sigma} : H^1(\Sigma_{\mathbb{C}}, T_{\Sigma_{\mathbb{C}}}) \to H^1(\Sigma_{\mathbb{C}}, T_{\Sigma_{\mathbb{C}}})$$
which is identified with the action of S on $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$ under the isomorphism
$$H^1(\Sigma_{\mathbb{C}}, T_{\Sigma_{\mathbb{C}}}) \cong \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X).$$

The pair $(T_{\Sigma_{\mathbb{C}}}, d\sigma)$ is the *holomorphic complex double* of the *Riemann-Hilbert bundle* $(T_\Sigma, T_{\partial\Sigma}) \to (\Sigma, \partial\Sigma)$, where a Riemann-Hilbert bundle and its holomorphic complex double are defined in [24, Section 3.3.4]. There is an isomorphism (see [24, Section 3.4])
$$H^1(\Sigma_{\mathbb{C}}, T_{\Sigma_{\mathbb{C}}})^{\tilde{\sigma}} \cong H^1(\Sigma, \partial\Sigma, T_\Sigma, T_{\partial\Sigma}).$$

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From the above discussion, we know that $H^1(\Sigma_{\mathbb{C}}, T_{\Sigma_{\mathbb{C}}})^{\tilde{\sigma}}$ gives the first order deformation of the symmetric Riemann surfaces $(\Sigma_{\mathbb{C}}, \sigma)$. Given a differential family of symmetric Riemann surface $(\tilde{\Sigma}_t, \sigma_t)$ such that $(\tilde{\Sigma}_0, \sigma_0) = (\Sigma_{\mathbb{C}}, \sigma)$, $K_t = \tilde{\Sigma}_t/\langle\sigma_t\rangle$ is a family of Klein surfaces. Each K_t is homeomorphic to Σ, which is orientable, so it admits two analytic structures compatible with its dianalytic structure, and one is the complex conjugate of the other. We get two differentiable families $M_t, M_t' = \bar{M}_t$ of bordered Riemann surfaces, where M_t is a deformation of Σ, and the other is a deformation of $\bar{\Sigma}$. Therefore, $H^1(\Sigma, \partial\Sigma, T_\Sigma, T_{\partial\Sigma})$ should give infinitesimal deformations of Σ.

Recall that an infinitesimal deformation of $\Sigma_{\mathbb{C}}$ determines a Čech 1 cocycle in $H^1(\tilde{\mathcal{A}}, \Theta_{\mathbb{C}}) \subset H^1(\Sigma_{\mathbb{C}}, T_{\Sigma_{\mathbb{C}}})$, where $\tilde{\mathcal{A}}$ is an analytic atlas of $\Sigma_{\mathbb{C}}$, and $\Theta_{\mathbb{C}}$ is the sheaf of local holomorphic vector fields on $\Sigma_{\mathbb{C}}$ [23]. (The inclusion is an isomorphism if $\tilde{\mathcal{A}}$ is acyclic). Following argument similar to that in [23], we now show that an infinitesimal deformation of Σ determines a Čech 1 cocycle in $H^1(\mathcal{A}, \Theta_\Sigma)$, where $\mathcal{A}$ is an analytic atlas of Σ, and Θ_Σ is the sheaf of local holomorphic vector fields on T_Σ with boundary values in $T_{\partial\Sigma}$.

Let $\{M_t \mid t \in B\}$ be a differentiable family of bordered Riemann surfaces, $M_0 = \Sigma$. We use the notation in Definition 3.3. Then

$$\mathcal{A} = \{(U_i, \phi_i) \equiv \left(\mathcal{U}_i \cap \pi^{-1}(0), z_i|_{\mathcal{U}_i \cap \pi^{-1}(t)}\right) \mid i \in I, \mathcal{U}_i \cap \pi^{-1}(t) \neq \phi\}$$

is an analytic atlas of Σ. Without loss of generality, we may assume that $\mathcal{A}$ is acyclic. We define t-dependent transition functions f_{ij} by $z_i = f_{ij}(z_j, t) = f_{ik}(z_k, t)$. Then $f_{ij}(z_j, t) \in \mathbb{R}$ if $z_j \in \mathbb{R}$ by part 4 of Definition 3.3.

$$\begin{aligned} z_i &= f_{ik}(z_k, t) = f_{ij}(f_{jk}(z_k, t), t) \\ \frac{\partial f_{ik}}{\partial t} &= \frac{\partial f_{ij}}{\partial z_j}\frac{\partial f_{jk}}{\partial t} + \frac{\partial f_{ij}}{\partial t} \end{aligned}$$

Multiplying by $\frac{\partial}{\partial z_i}$ and noting that $\frac{\partial f_{ij}}{\partial z_j} = \frac{\partial z_i}{\partial z_j}$, we have

$$\frac{\partial f_{ik}}{\partial t}\frac{\partial}{\partial z_i} = \frac{\partial f_{jk}}{\partial t}\frac{\partial}{\partial z_j} + \frac{\partial f_{ij}}{\partial t}\frac{\partial}{\partial z_i}.$$

Let (x_i, y_i) be real coordinates defined by $z_i = x_i + iy_i$, then the boundary is defined by $\{y_i = 0\}$, and the tangent line to the boundary is spanned by $\frac{\partial}{\partial x_i}$. Under the isomorphism $T_\Sigma \to T_\Sigma^{0,1}$ given by $v \mapsto (v - iJv)/2$, we have $\frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial z}$ and $\frac{\partial}{\partial y} \mapsto i\frac{\partial}{\partial z}$. So $\theta_{ik} \equiv \left.\frac{\partial f_{ik}}{\partial t}\right|_{t=0} \frac{\partial}{\partial z_i}$ defines a Čech 1 cochain in $C^1(\mathcal{A}, \Theta_\Sigma)$ which satisfies the cocycle condition $\theta_{ik} = \theta_{ij} + \theta_{jk}$.

By exactly the same argument as in [23] we see that another system of coordinates will give rise to a Čech 1 cocycle $\theta' = \theta + \delta\alpha$, where α is a Čech 0 cochain. Therefore, the infinitesimal deformation of Σ is given by

$$H^1(\mathcal{A}, \Theta_\Sigma) = H^1(\Sigma, \partial\Sigma, T_\Sigma, T_{\partial\Sigma}).$$

3.2. **Nodal bordered Riemann surfaces.** To compactify the moduli of bordered Riemann surfaces, we will allow nodal singularities. The complex double of a bordered Riemann surface is a complex algebraic curve with real structure. The stable compactification of moduli of such curves parametrizes

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stable complex algebraic curves with real structure [36,41], or equivalently, stable compact symmetric Riemann surfaces. Naively, the quotient of a stable compact symmetric Riemann surface by its antiholomorphic involution will give rise to a "stable bordered Riemann surface". We will make this idea precise in this section. This leads to the moduli space $\overline{M}_{(g,h)}$ of stable bordered Riemann surfaces of type (g,h), and the moduli space $\overline{M}_{(g,h),(n,\vec{m})}$ of stable marked bordered Riemann surfaces of type (g,h) with $(n,\vec{m})$ marked points.

Let Σ be a (smooth) bordered Riemann surface of type (g,h). Note that if $\phi:\Sigma\to\Sigma$ is an automorphism (Definition 2.14), then its *complex double* ([24, Section 3.3.2]) $\phi_{\mathbb{C}}:\Sigma_{\mathbb{C}}\to\Sigma_{\mathbb{C}}$ is an automorphism of $(\Sigma_{\mathbb{C}},\sigma)$ (Definition 2.7). This gives an inclusion $\mathrm{Aut}(\Sigma)\subset\mathrm{Aut}(\Sigma_{\mathbb{C}},\sigma)$. It is easy to see that the following are equivalent:

- Σ is stable, i.e., $\mathrm{Aut}(\Sigma)$ is finite.
- $\Sigma_{\mathbb{C}}$ is stable.
- The genus $\tilde{g}=2g+h-1$ of $\Sigma_{\mathbb{C}}$ is greater than one.
- The Euler characteristic $\chi(\Sigma)=2-2g-h$ of Σ is negative.

We start with $\tilde{g}=2$. Let $\overline{M}_2$ be the moduli of stable complex algebraic curves of genus 2. The strata of $\overline{M}_2$ are shown in Figure 1.

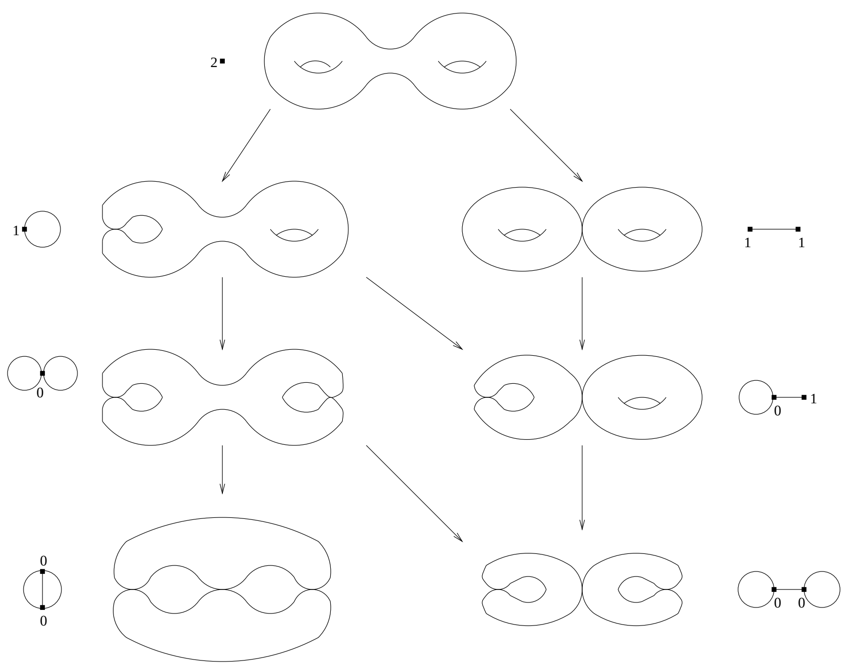

FIGURE 1. strata of $\overline{M}_2$. The dual graph and the underlying topological surface of a prestable curve are shown. In the dual graph of the curve C, each vertex corresponds to an irreducible component of C, labeled by the genus of the normalization, while each edge corresponds to a node of C, whose two end points correspond to the two irreducible components which intersect at this node.

If $\tilde{g}=2$, (g,h) can be $(0,3)$ (Figure 2) or $(1,1)$ (Figure 3).

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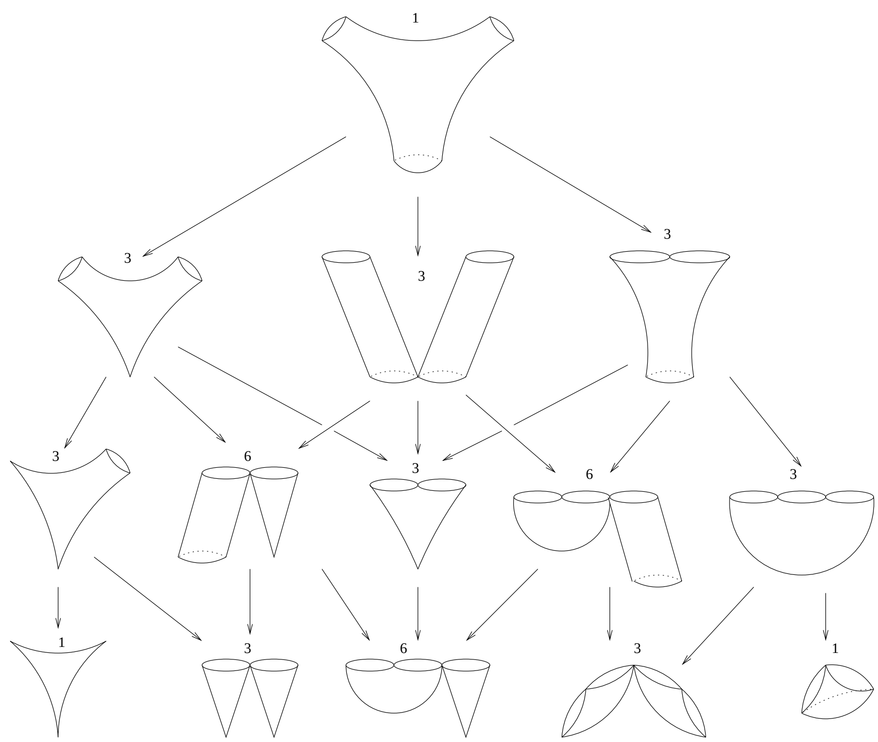

FIGURE 2. strata of $\overline{M}_{(0,3)}$. Each stable bordered Riemann surface above represents a topological type, and the number above each surface is the number of the associated strata. The strata associated to the same topological type are related by relabelling the three boundary circles. There are one 3-dimensional stratum, nine 2-dimensional strata, twenty-one 1-dimensional strata, and fourteen 0-dimensional strata. We will see in Example 4.7 that $\overline{M}_{(0,3)}$ can be identified with the associahedron K_5 defined by J. Stasheff [40].

Definition 3.4. *Let (x, y) be coordinates on $\mathbb{C}^2$, and $A(x, y) = (\bar{x}, \bar{y})$ be the complex conjugation. A* node *on a bordered Riemann surface is a singularity isomorphic to one of the following:*

(1) $(0, 0) \in \{xy = 0\}$ *(interior node)*
(2) $(0, 0) \in \{x^2 + y^2 = 0\}/A$ *(boundary node of type E)*
(3) $(0, 0) \in \{x^2 - y^2 = 0\}/A$ *(boundary node of type H)*

A nodal bordered Riemann surface *is a singular bordered Riemann surface whose singularities are nodes.*

A type E boundary node on a bordered Riemann surface corresponds to a boundary component shrinking to a point, while a type H boundary node corresponds to a boundary component intersecting itself or another boundary component. The boundary of a nodal Riemann surface is a union of points and circles, where one circle might intersect other circles in finitely many points.

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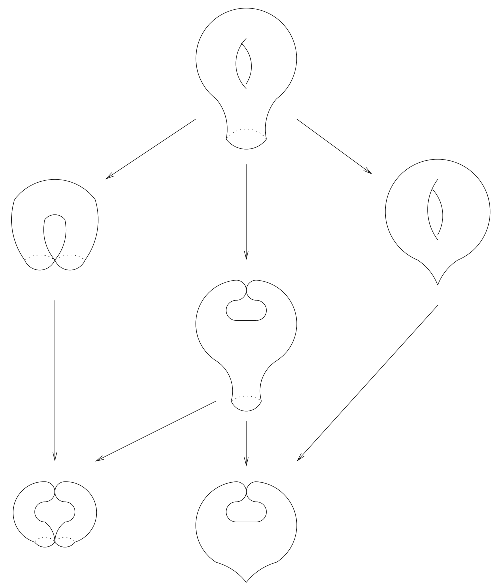

FIGURE 3. strata of $\overline{M}_{(1,1)}$

The notion of morphisms and complex doubles can be easily extended to nodal bordered Riemann surfaces. The complex double of a nodal bordered Riemann surface is a nodal compact symmetric Riemann surface.

Example 3.5. *Consider $C_\epsilon = \{X^2 - Y^2 + \epsilon Z^2 = 0\} \subset \mathbf{P}^2$, where $[X, Y, Z]$ are homogeneous coordinates on $\mathbf{P}^2$, and $\epsilon \in \mathbb{R}$. C_ϵ is invariant under the standard complex conjugation $A([X, Y, Z]) = [\bar{X}, \bar{Y}, \bar{Z}]$ on $\mathbf{P}^2$, so $\sigma_\epsilon = A|_{C_\epsilon}$ is an antiholomorphic involution on C_ϵ. For $\epsilon \neq 0$, $(C_\epsilon, \sigma_\epsilon)$ is a symmetric Riemann surface of type $(0, 1, 0)$, and $C_\epsilon/\langle\sigma_\epsilon\rangle$ is the disc, which is a bordered Riemann surface. C_0 has two irreducible components $\{X + Y = 0\}$ and $\{X - Y = 0\}$ which are projective lines, and the intersection point $[0, 0, 1]$ of the two lines is a node on C_0. Both lines are invariant under the antiholomorphic involution σ_0. $C_0/\langle\sigma_0\rangle$ is a nodal bordered Riemann surface: it is the union of two discs whose intersection is a boundary node of type H.*

Example 3.6. *Consider $C_\epsilon = \{X^2 + Y^2 + \epsilon Z^2 = 0\} \subset \mathbf{P}^2$, where $\epsilon \in \mathbb{R}$. C_ϵ is invariant under the complex conjugation A on $\mathbf{P}^2$, so $\sigma_\epsilon = A|_{C_\epsilon}$ is an antiholomorphic involution on C_ϵ. Set $\Sigma_\epsilon = C_\epsilon/\langle\sigma_\epsilon\rangle$. For $\epsilon > 0$, $(C_\epsilon, \sigma_\epsilon)$ is a symmetric Riemann surface of type $(0, 0, 1)$, and Σ_ϵ is the real projective plane; for $\epsilon < 0$, $(C_\epsilon, \sigma_\epsilon)$ is a symmetric Riemann surface of type $(0, 1, 0)$, and Σ_ϵ is the disc. C_0 has two irreducible components $\{X + \sqrt{-1}Y = 0\}$ and $\{X - \sqrt{-1}Y = 0\}$ which are projective lines, and their intersection point $[0, 0, 1]$ is a node on C_0. The antiholomorphic involution σ_0 interchanges the two irreducible components of C_0 and leaves the node invariant, thus $\Sigma_0 \cong \mathbf{P}^1$, which is a smooth Riemann surface without boundary. However, we would like to view it as a disc whose boundary shrinks to a point which is a boundary node of type E.*

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Definition 3.7. *Let Σ be a nodal bordered Riemann surface. The antiholomorphic involution σ on its complex double $\Sigma_{\mathbb{C}}$ can be lifted to $\hat{\sigma}: \widehat{\Sigma_{\mathbb{C}}} \to \widehat{\Sigma_{\mathbb{C}}}$, where $\widehat{\Sigma_{\mathbb{C}}}$ is the normalization of $\Sigma_{\mathbb{C}}$ (viewed as a complex algebraic curve). The* normalization *of $\Sigma = \Sigma_{\mathbb{C}}/\langle\sigma\rangle$ is defined by $\hat{\Sigma} = \widehat{\Sigma_{\mathbb{C}}}/\langle\hat{\sigma}\rangle$.*

From the above definition, the complex double of the normalization is the normalization of the complex double, i.e., $\hat{\Sigma}_{\mathbb{C}} = \widehat{\Sigma_{\mathbb{C}}}$.

Let Σ be a smooth bordered Riemann surfaces of type (g, h). The following are possible degenerations of Σ whose only singularity is a boundary node.

E. One boundary component shrinks to a point. The normalization is a smooth bordered Riemann surface of type $(g, h-1)$ (Figure 4).

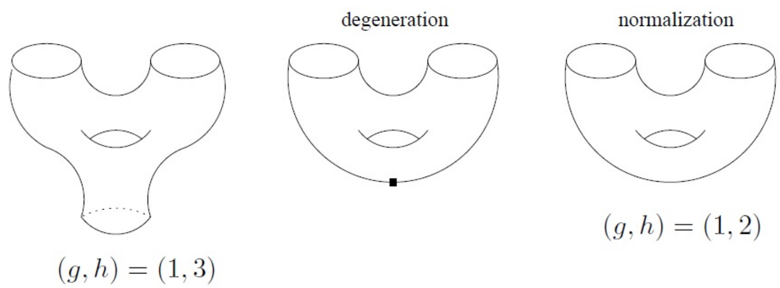

FIGURE 4. boundary node of type E

H1. Two boundary components intersect at one point. The normalization is a smooth bordered Riemann surface of type $(g, h-1)$, and the two preimages of the node are on the same boundary component (Figure 5).

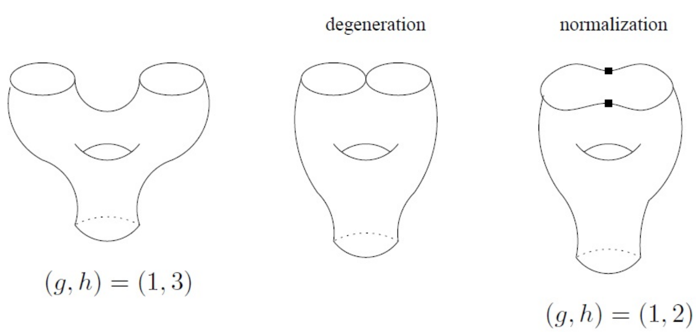

FIGURE 5. boundary node of type H1

H2. One boundary component intersects itself, and the normalization of the surface is connected. The normalization is a smooth bordered Riemann surface of type $(g-1, h+1)$, and the two preimages of the node are on different boundary components (Figure 6).

H3. One boundary component intersects itself, and the normalization of the surface is disconnected. The normalization is a disjoint union of two smooth bordered Riemann surfaces of types (g_1, h_1)

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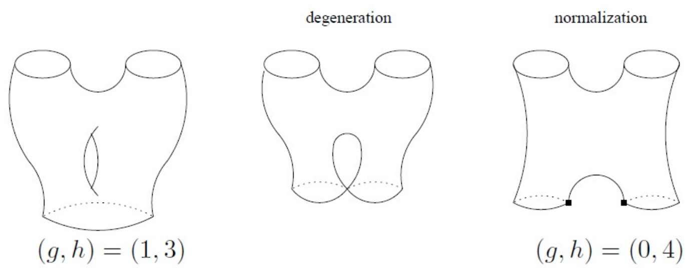

FIGURE 6. boundary node of type H2

and (g_2, h_2) such that $g = g_1 + g_2$ and $h = h_1 + h_2 - 1$, and each connected component contains one of the two preimages of the node (Figure 7).

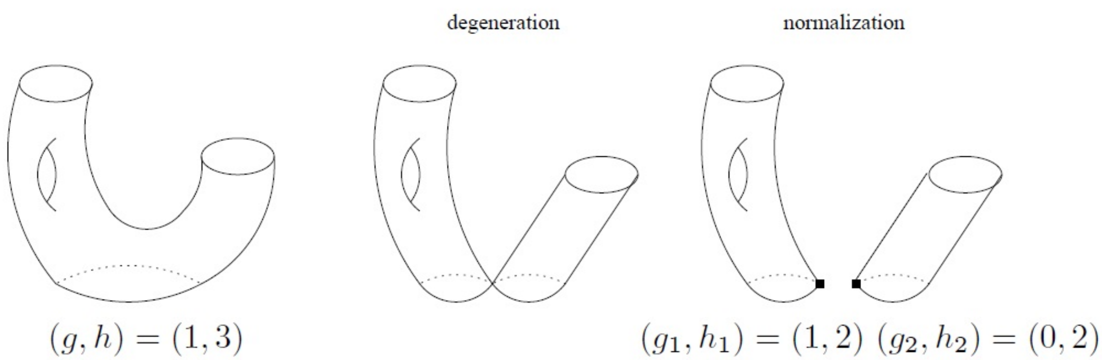

FIGURE 7. boundary node of type H3

Definition 3.8. *A* prestable *bordered Riemann surface is either a smooth bordered Riemann surface or a nodal bordered Riemann surface.*

Let Σ be a prestable bordered Riemann surface, $\hat{\Sigma}$ be its normalization. Let $\hat{C}_1, \ldots, \hat{C}_\nu$, $\hat{\Sigma}_1, \ldots, \hat{\Sigma}_{\nu'}$ be the connected components of $\hat{\Sigma}$, where $\hat{C}_i$ is a smooth Riemann surface of genus $\hat{g}_i$, and $\hat{\Sigma}_{i'}$ is a smooth bordered Riemann surface of type $(g_{i'}, h_{i'})$. Let δ be the number of connecting interior nodes (Figure 8), and $\delta_E, \delta_{H1}, \delta_{H2}, \delta_{H3}$ be the numbers of boundary nodes described in $E, H1, H2, H3$, respectively.

The topological type (g, h) of Σ is given by

$$\begin{aligned} g &= \tilde{g}_1 + \cdots + \tilde{g}_\nu + g_1 + \cdots + g_{\nu'} + \delta + \delta_{H2} \\ h &= h_1 + \cdots + h_{\nu'} + \delta_E + \delta_{H1} - \delta_{H2} - \delta_{H3} \end{aligned}$$

It is now straightforward to extend the notion of marked bordered Riemann surfaces to prestable bordered Riemann surfaces.

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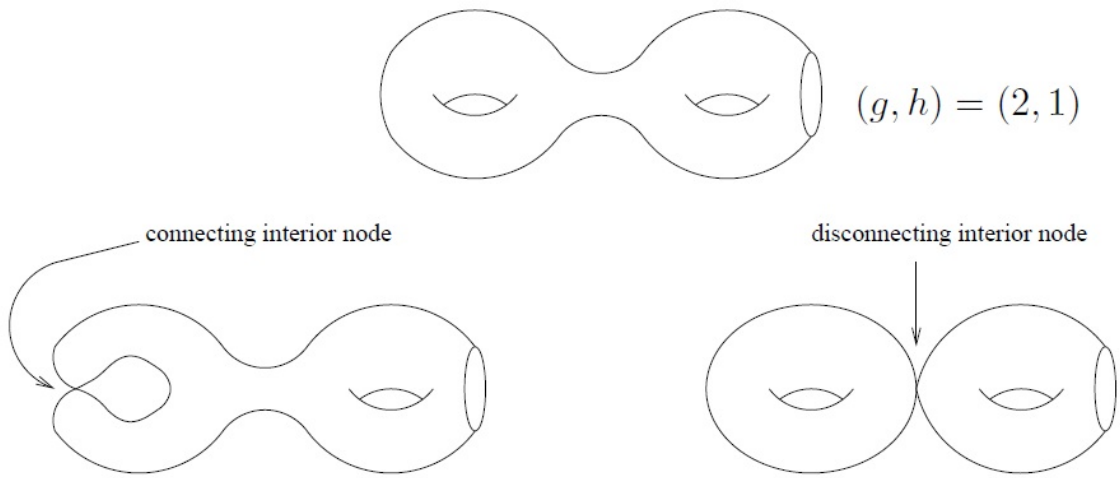

FIGURE 8. connecting and disconnecting nodes

Definition 3.9. *Let h be a positive integer, g,n be nonnegative integers, and $\vec{m}=(m^1,\ldots,m^h)$ be an h-uple of nonnegative integers. A* prestable marked bordered Riemann surface of type (g,h) with $(n,\vec{m})$ marked points *is an $(h+3)$-uple*

$$(\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h)$$

whose components are described as follows.

- *Σ is a prestable bordered Riemann surface of type (g,h).*
- *$\mathbf{B}=(B^1,\ldots,B^h)$, where $\partial\Sigma=\bigcup_{i=1}^h B^i$, and each B^i is an immersed circle. The circles $B^1,\ldots,B^h$ may intersect each other at boundary nodes, and become h disjoint embedded circles under smoothing of all boundary nodes.*
- *$\mathbf{p}=(p_1,\ldots,p_n)$ is an n-uple of distinct smooth points in Σ°.*
- *$\mathbf{q}^i=(q^i_1,\ldots,q^i_{m^i})$ is an m^i-uple of distinct smooth points on B^i.*

Definition 3.10. *A* morphism *between prestable marked bordered Riemann surfaces of type (g,h) with $(n,\vec{m})$ marked points*

$$(\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h) \to (\Sigma', \mathbf{B}'; \mathbf{p}'; (\mathbf{q}')^1, \ldots, (\mathbf{q}')^h)$$

is an isomorphism of prestable bordered Riemann surfaces $\phi:\Sigma\to\Sigma'$ such that $\phi(B^i)=(B')^i$ for $i=1,\ldots,h$, $\phi(p_j)=p'_j$ for $j=1,\ldots,n$, and $\phi(q^i_k)=(q')^i_k$ for $k=1,\ldots,m^i$. A prestable marked bordered Riemann surfaces of type (g,h) with $(n,\vec{m})$ marked points is stable *if its automorphism group is finite.*

Remark 3.11. *The category of prestable marked bordered Riemann surfaces of type (g,h) with $(n,\vec{m})$ marked points is a groupoid since every morphism in Definition 3.10 is an isomorphism.*

Example 3.12. *Consider the case $(g,h)=(0,2), n=0, \vec{m}=(1,0)$ (annuli with one boundary marked point). The moduli space $\overline{M}_{(0,2)(0,(1,0))}$ is an interval $[0,1]$. There are three strata: $t\in(0,1), t=0, t=1$ (Figure 9).*

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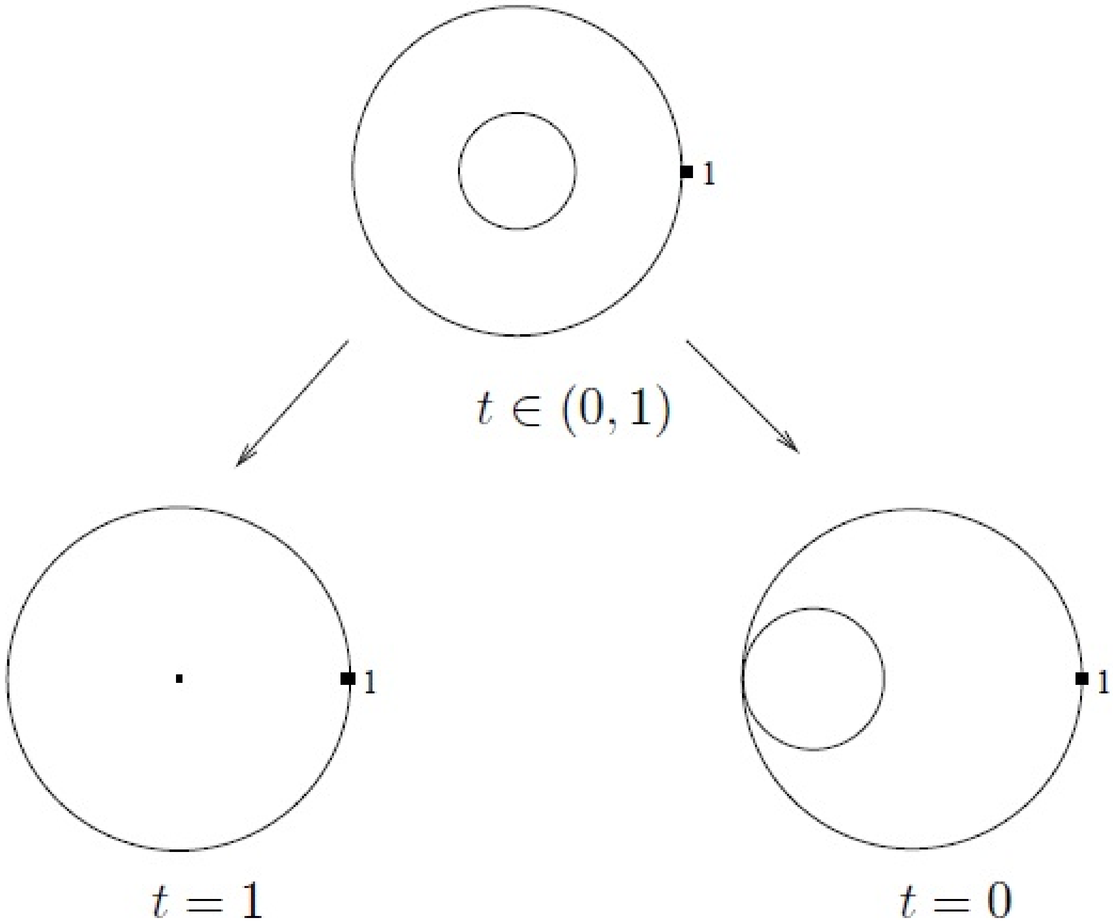

FIGURE 9. strata of $\overline{M}_{(0,2)(0,(1,0))}$

Example 3.13. *Consider the case* $(g,h) = (0,2), n = 0, \vec{m} = (2,0)$ (*annuli with two boundary marked points on the same boundary circle*). *The moduli space* $\overline{M}_{(0,2)(0,(2,0))}$ *is a pentagon. There are eleven strata* (*Figure* 10).

Example 3.14. *Consider the case* $(g,h) = (0,2), n = 0, \vec{m} = (1,1)$ (*annuli with one marked point on each boundary circle*). *The moduli space* $\overline{M}_{(0,2)(0,(1,1))}$ *is a disc* $\{z \in \mathbb{C} \mid |z| \leq 1\}$. *There are four strata:* $0 < |z| < 1$, $|z| = 1$ *but* $z \neq 1$, $z = 1$, $z = 0$ (*Figure* 11).

3.3. **Deformation theory for prestable bordered Riemann surfaces.** The algebraic approach of deformation theory for smooth bordered Riemann surfaces in Section 3.1 can be easily extended to nodal bordered Riemann surfaces. We will also consider marked points.

Let $(\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h)$ be a marked prestable bordered Riemann surface of type (g,h) with $(n, \vec{m})$ marked points. We want to study its infinitesimal deformation. The ordering of the boundary circles is irrelevant to the infinitesimal deformation, so in this section we will ignore it and write $(\Sigma; \mathbf{p}; \mathbf{q})$, where

$$\mathbf{q} = (q_1, \ldots, q_m) = (q_1^1, \ldots, q_{m^1}^1, q_1^2, \ldots, q_{m^2}^2, \ldots, q_1^h, \ldots, q_{m^h}^h),$$

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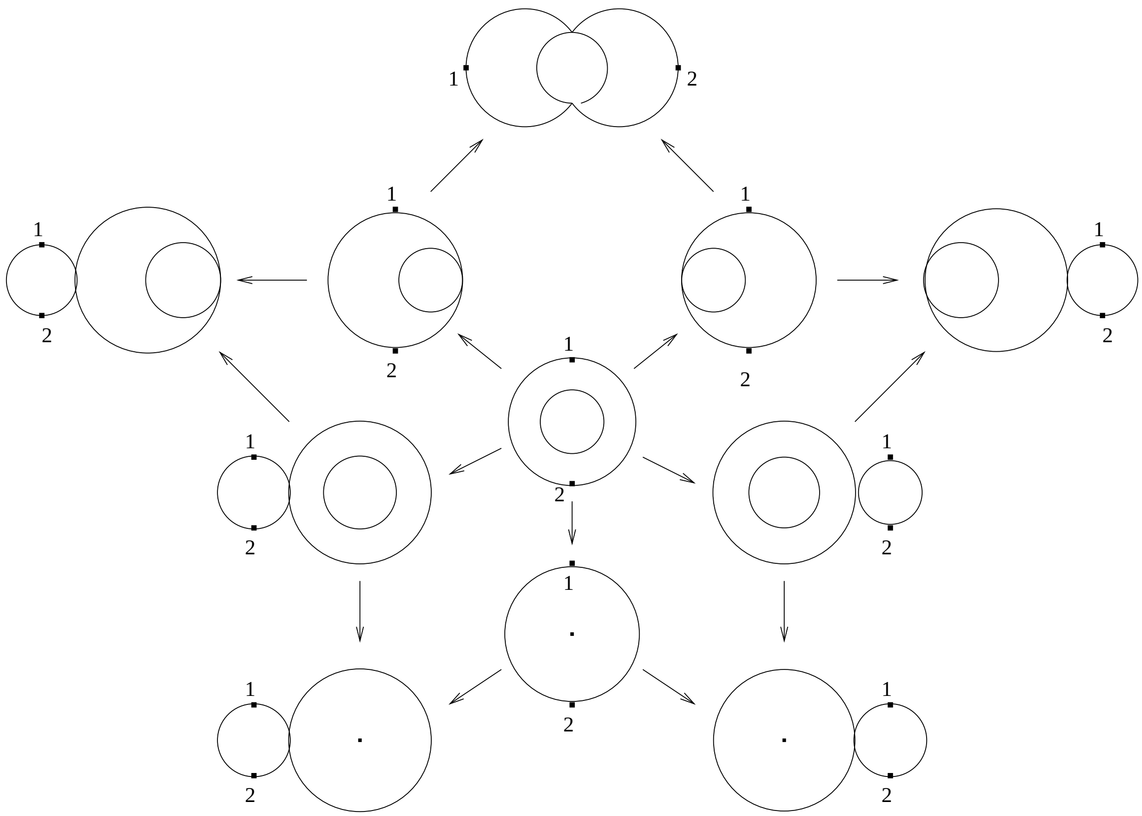

FIGURE 10. strata of $\overline{M}_{(0,2)(0,(2,0))}$

and $m = m^1 + \cdots + m^h$.

The complex double $(\Sigma_{\mathbb{C}}, \sigma)$ of Σ is a nodal complex algebraic curve X which is the complexification of some real algebraic curve X_0, i.e., $X = X_0 \times_{\mathbb{R}} \mathbb{C}$. Alternatively, (X, S) is a complex algebraic curve with a real structure, where S is a semi-linear automorphism which induces the antiholomorphic involution σ on $\Sigma_{\mathbb{C}}$.

Algebraically, the complex double of $(\Sigma; \mathbf{p}; \mathbf{q})$ is a nodal complex algebraic curve with $(2n+m)$ marked points $(X, \mathbf{x})$, where

$$\mathbf{x} = (x_1, \ldots, x_{2n+m}) = (p_1, \ldots, p_n, \bar{p}_1, \ldots, \bar{p}_n, q_1, \ldots, q_m).$$

Here we identify Σ with the image under the inclusion $i : \Sigma \to \Sigma_{\mathbb{C}}$, and denote $\sigma(p)$ by $\bar{p}$.

Let

$$D_{\mathbf{x}} = x_1 + \cdots + x_{2n+m}$$

be the divisor in X associated to $\mathbf{x}$. The set of first order deformation of the pointed complex algebraic curve $(X, \mathbf{x})$ is canonically identified with the complex vector space

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X),$$

and the obstruction lies in

$$\mathrm{Ext}^2_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X).$$

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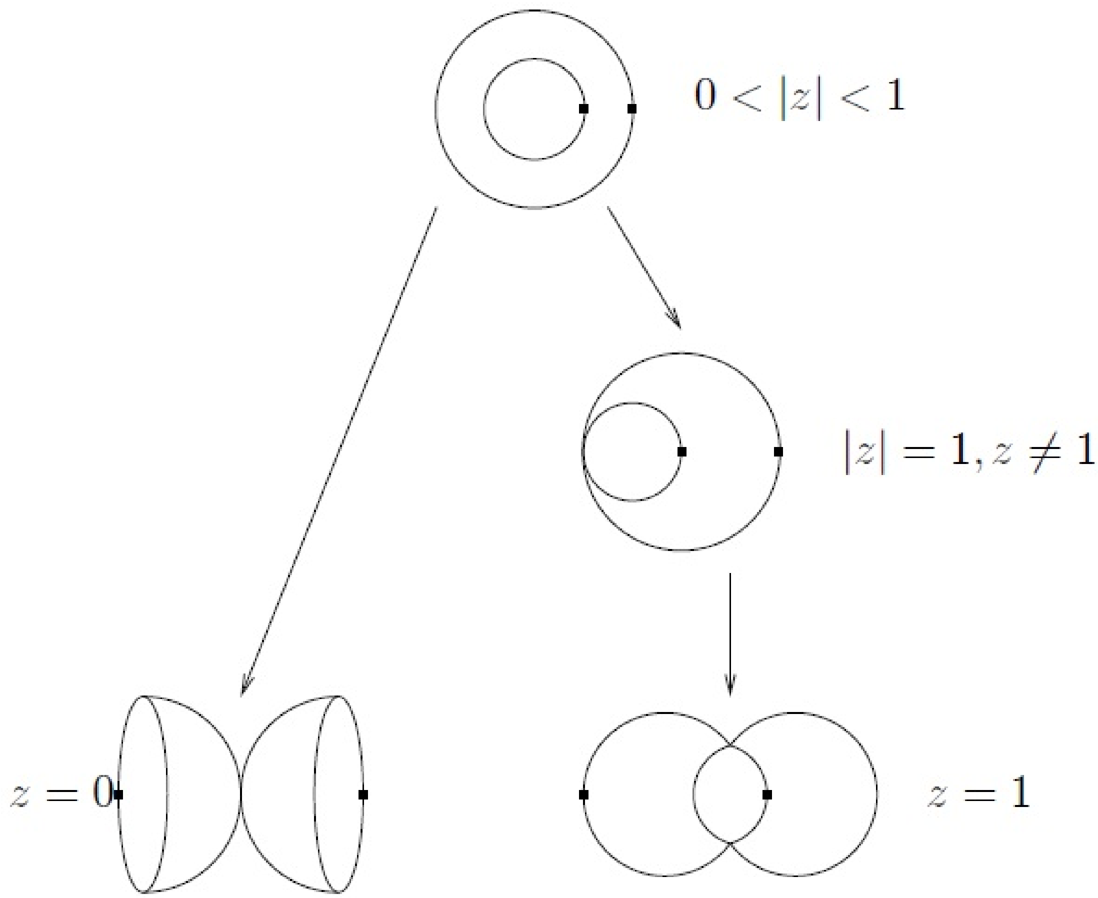

FIGURE 11. strata of $\overline{M}_{(0,2)(0,(1,1))}$

We claim that $\mathrm{Ext}^2_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X) = 0$. Three terms in the local to global spectrum sequence contribute to $\mathrm{Ext}^2_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X)$:

$$H^0(X, \mathcal{E}xt^2_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X)),$$
$$H^1(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X)),$$
$$H^2(X, \mathcal{E}xt^0_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X)).$$

The curve X has only nodal singularities, hence

$$\mathcal{E}xt^2_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X) = 0,$$

thus

$$H^0(X, \mathcal{E}xt^2_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X)) = 0.$$

The sheaf $\mathcal{E}xt^1_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X)$ is supported on nodes, so

$$H^1(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X)) = 0.$$

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Finally,

$$H^2(X, \mathcal{E}xt^0_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X)) = 0$$

since X is one dimensional. In this way we get the desired vanishing.

The semi-linear automorphism S induces a complex conjugation

$$S : \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X).$$

The fixed locus $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X)^S$ gives the first order deformation of $(X, \mathbf{x}, S)$ as a pointed complex algebraic curve with a real structure.

We will study the group $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X)$ and the action of S on it more closely. Let us first introduce some notation.

Let $C_1, ..., C_\nu$ be the irreducible components of Σ which are (possibly nodal) Riemann surfaces, and let $\Sigma_1, ..., \Sigma_{\nu'}$ be the remaining irreducible components of Σ, which are (possibly nodal) bordered Riemann surfaces. Then the irreducible components of X are

$$C_1, \ldots, C_\nu,\ \bar{C}_1, \ldots, \bar{C}_\nu,\ (\Sigma_1)_{\mathbb{C}}, \ldots, (\Sigma_{\nu'})_{\mathbb{C}}.$$

Let $\hat{C}_i$ denote the normalization of C_i, $i = 1, \ldots, \nu$, and let $\hat{\Sigma}_{i'}$ denote the normalization of $\Sigma_{i'}$, $i' = 1, \ldots, \nu'$. Then

$$\hat{C}_1, \ldots, \hat{C}_\nu,\ \hat{\Sigma}_1, \ldots, \hat{\Sigma}_{\nu'}$$

are the connected components of $\hat{\Sigma}$, the normalization of Σ, and

$$\hat{C}_1, \ldots, \hat{C}_\nu,\ \hat{\bar{C}}_1, \ldots, \hat{\bar{C}}_\nu,\ (\hat{\Sigma}_1)_{\mathbb{C}}, \ldots, (\hat{\Sigma}_{\nu'})_{\mathbb{C}}.$$

are the connected components of $\hat{X}$, the normalization of X.

Let $r_1, \ldots, r_{l_0} \in \Sigma^\circ$ be interior nodes of Σ, and $s_1, \ldots, s_{l_1} \in \partial\Sigma$ be boundary nodes of Σ. Then X has $2l_0 + l_1$ nodes $r_1, \ldots, r_{l_0},\ \bar{r}_1, \ldots, \bar{r}_{l_0},\ s_1, \ldots, s_{l_1}$.

Let $\hat{p}_j \in \hat{\Sigma}$ be the preimage of p_j under the normalization map $\pi : \hat{X} \to X, j = 1, \ldots, n$. Define $\hat{\bar{p}}_j, \hat{q}_{j'}$ similarly. Let $\hat{r}_\alpha, \hat{r}_{l_0+\alpha}$ be the preimages of $r_\alpha, \alpha = 1, \ldots, l_0$, and define $\hat{s}_{\alpha'}, \hat{s}_{l_1+\alpha'}$ similarly.

Consider $\hat{X}$ with marked points

$$\hat{\mathbf{x}} = (\hat{p}_1, \ldots, \hat{p}_n,\ \hat{\bar{p}}_1, \ldots, \hat{\bar{p}}_n,\ \hat{q}_1, \ldots, \hat{q}_m,\ \hat{r}_1, \ldots, \hat{r}_{2l_0},\ \hat{\bar{r}}_1, \ldots, \hat{\bar{r}}_{2l_0},\ \hat{s}_1, \ldots, \hat{s}_{2l_1}),$$

which can be written as a disjoint union of pointed curves

$$(\hat{C}_i, \mathbf{y}^i),\quad (\hat{\bar{C}}_i, \bar{\mathbf{y}}^i),\quad ((\hat{\Sigma}_{i'})_{\mathbb{C}}, \mathbf{z}^{i'}) = ((\hat{\Sigma}_{i'})_{\mathbb{C}}, \mathbf{p}^{i'}, \bar{\mathbf{p}}^{i'}, \mathbf{q}^{i'}),$$

where

$$\begin{aligned} &\mathbf{y}^i = (y^i_1, \ldots, y^i_{\tilde{n}_i}),\quad \bar{\mathbf{y}}^i = (\bar{y}^i_1, \ldots, \bar{y}^i_{\tilde{n}_i}),\\ &\mathbf{p}^{i'} = (p^{i'}_1, \ldots, p^{i'}_{n_{i'}}),\quad \bar{\mathbf{p}}^{i'} = (\bar{p}^{i'}_1, \ldots, \bar{p}^{i'}_{n_{i'}}),\quad \mathbf{q}^{i'} = (q^{i'}_1, \ldots, q^{i'}_{m_{i'}}), \end{aligned}$$

$i = 1, \ldots, \nu$, $i' = 1, \ldots, \nu'$, $p^{i'}_1, \ldots, p^{i'}_{n_{i'}} \in \hat{\Sigma}^\circ_{i'}$, $q^{i'}_1, \ldots, q^{i'}_{m_{i'}} \in \partial\hat{\Sigma}_{i'}$, and

$$\sum_{i=1}^{\nu} \tilde{n}_i + \sum_{i'=1}^{\nu'} n_i = n + 2l_0,\quad \sum_{i=1}^{\nu} m_i = m + 2l_1.$$

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From the local to global spectrum sequence, we have an exact sequence

$$\begin{aligned} 0 \quad \to \quad & H^1(X, \mathcal{E}xt^0_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X)) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X) \\ \to \quad & H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X)) \to 0, \end{aligned}$$

where

$$\mathcal{E}xt^0_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X) = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X) = \Omega^1_X(D_{\mathbf{x}})^\vee.$$

We have the following elementary fact:

Lemma 3.15.

$$\Omega^1_X(D_{\mathbf{x}})^\vee = \pi_* T_{\hat{X}}(-D_{\hat{\mathbf{x}}}),$$

where $\pi : \hat{X} \to X$ is the normalization map, and $D_{\hat{\mathbf{x}}}$ is the divisor corresponding to the marked points $\hat{\mathbf{x}}$ in $\hat{X}$.

Proof. The equality obviously holds for smooth points. It suffices to show that $\Omega^1_Y = \pi_* T_{\hat{Y}}$ for $Y = \mathrm{Spec}\mathbb{C}[x,y]/(xy)$, which follows by a local calculation. □

The map $\pi : \hat{X} \to X$ is an affine morphism, so by [16, Chapter III, Exercise 4.1] we have

$$H^1(X, \mathcal{E}xt^0_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X)) = H^1(X, \pi_* T_{\hat{X}}(-D_{\hat{\mathbf{x}}})) = \bigoplus_{i=1}^{\nu}(W_i \oplus \bar{W}_i) \oplus \bigoplus_{i'=1}^{\nu'} W'_{i'}$$

where the vector spaces

$$\begin{aligned} & W_i = H^1(\hat{C}_i, T_{\hat{C}_i}(-D_{\mathbf{y}^i})), \quad \bar{W}_i = H^1(\hat{\bar{C}}_i, T_{\hat{\bar{C}}_i}(-D_{\bar{\mathbf{y}}^i})), \\ & W'_{i'} = H^1((\hat{\Sigma}_{i'})_{\mathbb{C}}, T_{(\hat{\Sigma}_{i'})_{\mathbb{C}}}(-D_{\mathbf{z}^{i'}})) \end{aligned}$$

correspond to deformations of pointed curves $(\hat{C}_i, \mathbf{y}^i)$, $(\hat{\bar{C}}_i, \bar{\mathbf{y}}^i)$, $((\hat{\Sigma}_{i'})_{\mathbb{C}}, \mathbf{z}^{i'})$, respectively.

Another local calculation shows that

$$H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X)) \cong \bigoplus_{\alpha=1}^{l_0}(V_\alpha \oplus \bar{V}_\alpha) \oplus \bigoplus_{\alpha'=1}^{l_1} V'_{\alpha'} \ ,$$

where

$$V_\alpha = T_{\hat{r}_\alpha}\hat{X} \otimes T_{\hat{r}_{l_0+\alpha}}\hat{X}, \quad \bar{V}_\alpha = T_{\hat{\bar{r}}_\alpha}\hat{X} \otimes T_{\hat{\bar{r}}_{l_0+\alpha}}\hat{X}, \quad V'_{\alpha'} = T_{\hat{s}_{\alpha'}}\hat{X} \otimes T_{\hat{s}_{l_0+\alpha'}}\hat{X}$$

correspond to smoothing of the nodes $r_\alpha, \bar{r}_\alpha, s_{\alpha'}$ in X, respectively.

Now the action of S on $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X)$ is clear: it acts on $W_i \oplus \bar{W}_i$ and $V_\alpha \oplus \bar{V}_\alpha$ by $(a,b) \mapsto (\bar{b}, \bar{a})$, and it acts on $W'_{i'}$ and $V'_{\alpha'}$ by $a \mapsto \bar{a}$. The real vector space ${W'_{i'}}^S$ corresponds to deformation of the pointed symmetric Riemann surface $((\hat{\Sigma}_{i'})_{\mathbb{C}}, \mathbf{z}^{i'}, \sigma)$. We also have ${W'_{i'}}^S \cong \hat{W}_{i'}$, where

$$\hat{W}_{i'} = H^1(\hat{\Sigma}_{i'}, \partial\hat{\Sigma}_{i'}, T_{\hat{\Sigma}_{i'}}(-p_1^{i'} - \cdots - p_{n_{i'}}^{i'}), T_{\partial\hat{\Sigma}_{i'}}(-q_1^{i'} - \ldots - q_{m_{i'}}^{i'}))$$

corresponds to deformation of the pointed bordered Riemann surface $(\hat{\Sigma}_{i'}, \mathbf{p}^{i'}, \mathbf{q}^{i'})$.

The action of S on $V'_{\alpha'}$ can be understood by studying local models

$$\{x^2 + y^2 = 0\}/A \qquad (\text{type E}) \ ,$$

$$\{x^2 - y^2 = 0\}/A \qquad (\text{type H}) \ ,$$

where the antiholomorphic involution is $A(x,y)=(\bar{x},\bar{y})$. The deformation of $\{x^2 \pm y^2 = 0\}$ is given by $\{x^2 \pm y^2 = \epsilon\}$, where $\epsilon \in \mathbb{C}$ is small. A acts on deformation by $\epsilon \mapsto \bar{\epsilon}$, so the deformation of $(\{x^2 \pm y^2 = 0\}, A)$ is given by $(\{x^2 \pm y^2 = \epsilon\}, A)$, where $\epsilon \in \mathbb{R}$ is small. $\{x^2 \pm y^2 = \epsilon\}/A$, $\epsilon \in \mathbb{R}$ small gives a deformation of the boundary nodes, and there is a topological transition from $\epsilon > 0$ to $\epsilon < 0$ (Figure 12, 13, 14, 15). $\epsilon \in \mathbb{C}$ corresponds to $V'_{\alpha'}$, $\epsilon \in \mathbb{R}$ corresponds to ${V'_{\alpha'}}^S = \hat{V}_{\alpha'}$, and $\epsilon \geq 0$ corresponds to $\hat{V}^+_{\alpha'}$.

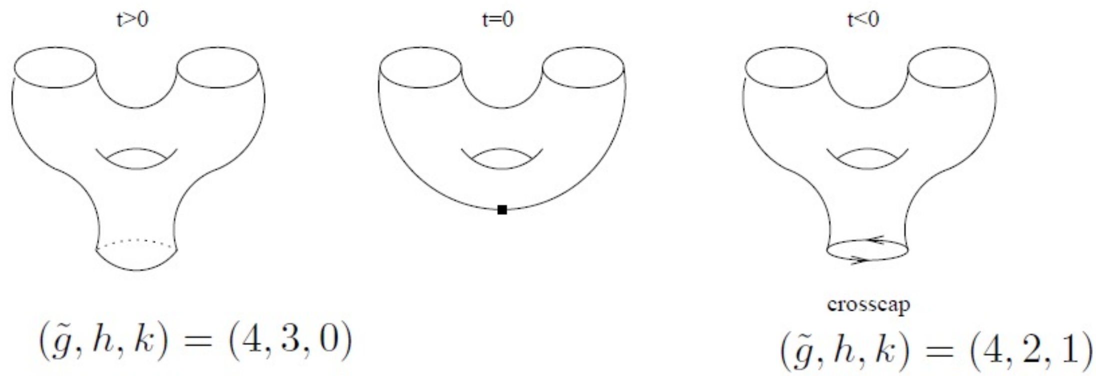

FIGURE 12. type E

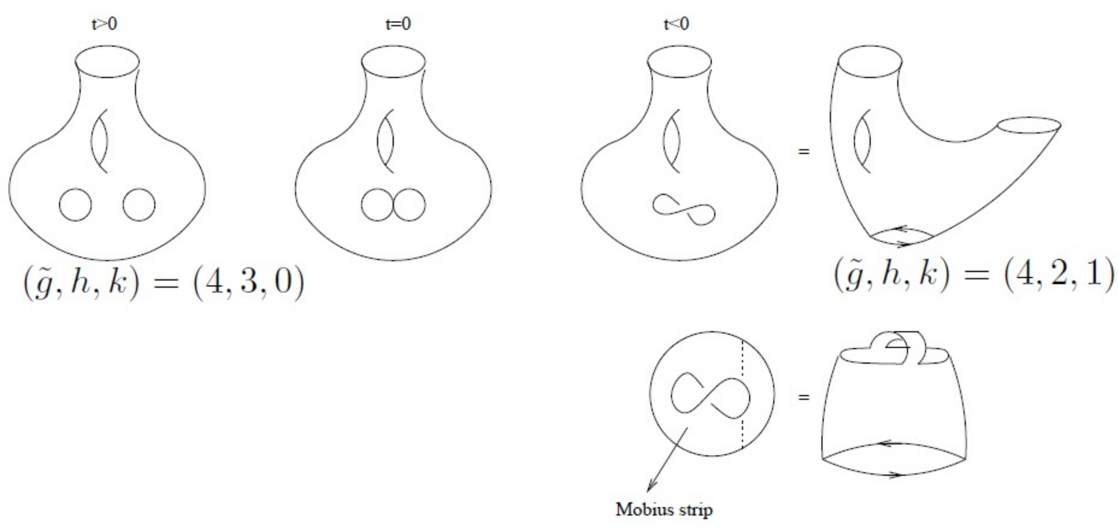

FIGURE 13. type H1

Above discussion can be summarized as follows.

(1) The infinitesimal deformation of the pointed complex algebraic curve $(X, \mathbf{x})$ is given by

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X) \cong \bigoplus_{i=1}^{\nu}(W_i \oplus \bar{W}_i) \oplus \bigoplus_{i'=1}^{\nu'} W'_{i'} \oplus \bigoplus_{\alpha=1}^{l_0}(V_\alpha \oplus \bar{V}_\alpha) \oplus \bigoplus_{\alpha'=1}^{l_1} V'_{\alpha'}.$$

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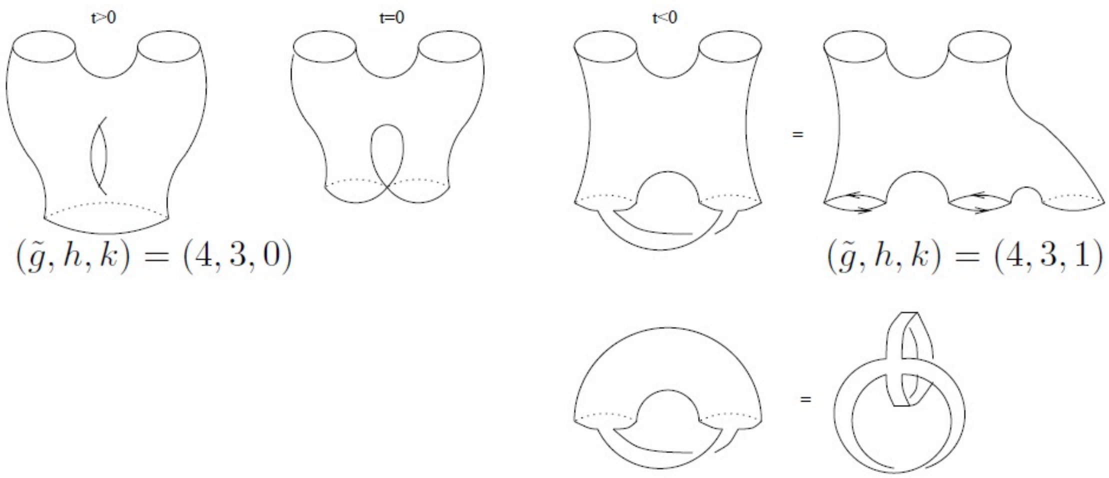

FIGURE 14. type H2

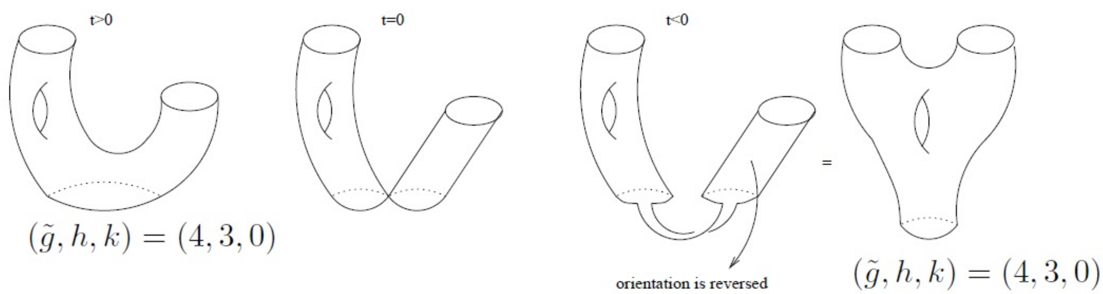

FIGURE 15. type H3

(2) The infinitesimal deformation of the pointed complex algebraic curve with a real structure $(X, \mathbf{x}, S)$ is given by

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X(D_{\mathbf{x}}), \mathcal{O}_X)^S \cong \bigoplus_{i=1}^{\nu}(W_i \oplus \bar{W}_i)^S \oplus \bigoplus_{i'=1}^{\nu'} {W'_{i'}}^S \oplus \bigoplus_{\alpha=1}^{l_0}(V_\alpha \oplus \bar{V}_\alpha)^S \oplus \bigoplus_{\alpha'=1}^{l_1} {V'_{\alpha'}}^S.$$

(3) The infinitesimal deformation of the pointed prestable bordered Riemann surface $(\Sigma; \mathbf{p}; \mathbf{q})$ is given by

$$\bigoplus_{i=1}^{\nu} W_i \oplus \bigoplus_{i'=1}^{\nu'} \hat{W}_{i'} \oplus \bigoplus_{\alpha=1}^{l_0} V_\alpha \oplus \bigoplus_{\alpha'=1}^{l_1} \hat{V}_\alpha^+.$$

4. Moduli of bordered Riemann surfaces

For stable complex algebraic curves, the moduli of complex structures can be identified with the moduli of hyperbolic structures. Under this identification, analytic methods are applicable to the study of moduli of stable curves. In [1], Abikoff constructed a topology on $\overline{M}_{g,n}$, the moduli of stable complex algebraic curves of genus g with n marked points, and showed that $\overline{M}_{g,n}$ is compact and Hausdorff in this topology. Similarly, Seppälä [36] constructed a topology for the moduli space of

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stable real algebraic curves of a given genus $g > 1$, and showed that the moduli space is compact and Hausdorff in the topology. In this section, we describe how these results can be modified to study stable bordered Riemann surfaces.

4.1. **Various moduli spaces and their relationships.** Let

$$(\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \dots, \mathbf{q}^h)$$

be a marked stable bordered Riemann surface of type (g, h) with $(n, \vec{m})$ marked points. The moduli of the ordering of the circles $B^1, \dots, B^h$ is given by the permutation group of h elements. Let $(\Sigma_{\mathbb{C}}, \sigma)$ be the complex double of Σ, and let

$$\mathbf{x} = (x_1, \dots, x_{2n+m}) = (p_1, \dots, p_n, \bar{p}_1, \dots, \bar{p}_n, q_1, \dots, q_m)$$

$(\Sigma_{\mathbb{C}}, \sigma, \mathbf{x})$ be a stable symmetric Riemann surface of genus $\tilde{g} = 2g + h - 1$ with (n, m) marked points. Removing $x_1, \dots, x_{\tilde{n}}$ from $\Sigma_{\mathbb{C}}$, where $\tilde{n} = 2n + m$, we obtain (S, σ), a stable symmetric Riemann surface of genus $\tilde{g}$ with $\tilde{n}$ punctures. Let S' be the complement of nodes in S. There is a one to one correspondence between connected components of S' and irreducible components of S. Each connected component of S' is a smooth punctured Riemann surface. The stability condition is equivalent to the statement that each connected component of S' has negative Euler characteristic. Therefore, there is a unique complete hyperbolic metric in the conformal class of Riemannian metrics on S' determined by the complex structure.

Let $\overline{M}_{\tilde{g},\tilde{n}}$ be the moduli of stable compact Riemann surfaces of genus $\tilde{g}$ with $\tilde{n}$ marked points, or equivalently, the moduli of stable complex algebraic curves of genus $\tilde{g}$ with $\tilde{n}$ marked points. Let $\bar{P}_{\tilde{g},\tilde{n}}$ be the moduli of stable oriented hyperbolic surfaces of genus $\tilde{g}$ with $\tilde{n}$ punctures. From the above discussion we know that there is a surjective map $\tilde{\pi} : \overline{M}_{\tilde{g},\tilde{n}} \to \bar{P}_{\tilde{g},\tilde{n}}$. $\tilde{\pi}$ is generically $\tilde{n}!$ to one since the marked points are ordered, while the punctures are not. The fiber over a point in $\bar{P}_{\tilde{g},\tilde{n}}$ represented by the surface S consists of less than $\tilde{n}!$ points if and only if there is an automorphism of S permuting its punctures.

Let $\overline{M}^{\mathbb{R}}_{\tilde{g},(n,m)}$ be the moduli of stable symmetric compact Riemann surface of genus $\tilde{g}$ with (n, m) points (see Definition 2.8), and let $M^{\mathbb{R}}_{(\tilde{g},h,k),(n,m)}$ be the moduli of smooth symmetric compact Riemann surface of type $(\tilde{g}, h, k)$ (see Section 2.3) with (n, m) marked points. Note that $M^{\mathbb{R}}_{(\tilde{g},0,k),(n,m)}$ is empty if $m > 0$. $M^{\mathbb{R}}_{(\tilde{g},h,k),(n,m)}$ are disjoint subsets of $\overline{M}^{\mathbb{R}}_{\tilde{g},(n,m)}$, and their closures $\overline{M}^{\mathbb{R}}_{(\tilde{g},h,k),(n,m)}$ (in the topology defined later) cover $\overline{M}^{\mathbb{R}}_{\tilde{g},(n,m)}$.

There is an involution $A : \overline{M}_{\tilde{g},\tilde{n}} \to \overline{M}_{\tilde{g},\tilde{n}}$, given by

$$[(\Sigma, x_1, \dots, x_{\tilde{n}})] \mapsto [(\bar{\Sigma}, \sigma(x_{n+1}), \dots, \sigma(x_{2n}), \sigma(x_1), \dots, \sigma(x_n), \sigma(x_{2n+1}), \dots, \sigma(x_{2n+m}))]$$

where $\sigma : \Sigma \to \bar{\Sigma}$ is the canonical anti-holomorphic map from Σ to its complex conjugate $\bar{\Sigma}$. Let $\overline{M}^{A}_{\tilde{g},\tilde{n}}$ denote the fixed locus of A. Then there is a surjective map $\overline{M}^{\mathbb{R}}_{g,(n,m)} \to \overline{M}^{A}_{\tilde{g},\tilde{n}}$, given by forgetting the symmetry σ. This map is generically injective. It fails to be injective exactly when the automorphism group of $(\Sigma, \mathbf{x})$ is larger than that of $(\Sigma, \sigma, \mathbf{x})$.

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Let $\bar{P}^{\mathbb{R}}_{\tilde{g},\tilde{n}}$ be the moduli of stable symmetric oriented hyperbolic surfaces of genus $\tilde{g}$ with $\tilde{n}$ punctures, and let $P^{\mathbb{R}}_{(\tilde{g},h,k),\tilde{n}}$ the moduli of smooth symmetric oriented hyperbolic surfaces of type $(\tilde{g},h,k)$ with $\tilde{n}$ punctures. $P^{\mathbb{R}}_{(\tilde{g},h,k),\tilde{n}}$ are disjoint subsets of $\bar{P}^{\mathbb{R}}_{\tilde{g},\tilde{n}}$, and their closures $\bar{P}^{\mathbb{R}}_{(\tilde{g},h,k),\tilde{n}}$ (in the topology to be defined later) cover $\bar{P}^{\mathbb{R}}_{\tilde{g},\tilde{n}}$.

There is an involution $A' : \bar{P}_{\tilde{g},\tilde{n}} \to \bar{P}_{\tilde{g},\tilde{n}}$, given by $[S] \mapsto [\bar{S}]$. There is a surjective map $\bar{P}^{\mathbb{R}}_{\tilde{g},\tilde{n}} \to \bar{P}^{A'}_{\tilde{g},\tilde{n}}$, given by forgetting the symmetry σ. This map is generically injective. It fails to be injective exactly when the automorphism group of S is larger than that of (S,σ).

We have the following commutative diagrams:

$$
\begin{array}{ccccccc}
\bar{M}_{\tilde{g},\tilde{n}} & \xrightarrow{A} & \bar{M}_{\tilde{g},\tilde{n}} & \quad & \bar{M}^{\mathbb{R}}_{\tilde{g},(n,m)} & \longrightarrow & \bar{M}^{A}_{\tilde{g},\tilde{n}} \\
{\scriptstyle\tilde{\pi}}\downarrow & & {\scriptstyle\tilde{\pi}}\downarrow & & {\scriptstyle\pi^{\mathbb{R}}}\downarrow & & {\scriptstyle\pi^{A}}\downarrow \\
\bar{P}_{\tilde{g},\tilde{n}} & \xrightarrow{A'} & \bar{P}_{\tilde{g},\tilde{n}} & & \bar{P}^{\mathbb{R}}_{\tilde{g},(n,m)} & \longrightarrow & \bar{P}^{A'}_{\tilde{g},\tilde{n}}
\end{array}
$$

where the generic fiber of $\pi^{\mathbb{R}}$ consists of $2^n n! m!$ points. The factor 2^n corresponds to the permutation of the two points in each of the n conjugate pairs. The factor $n!$ corresponds to the permutation of the n conjugate pairs. The factor m corresponds to the permutation of the m marked points fixed by the symmetry. Similarly, the generic fiber of π^A consists of $2^n n! m!$ points. For a generic point in $\bar{P}^{A'}_{\tilde{g},\tilde{n}}$, its preimage under $\tilde{\pi}$ consists of $\tilde{n}!$ points, but only $2^n n! m!$ lie in the fixed locus of A.

Neither $\pi^{\mathbb{R}}$ nor π^A is surjective because the number of punctures fixed by the symmetry can be any integer between 0 and $\tilde{n}$, not only m.

We are interested in the moduli space $\overline{M}_{(g,h),(n,\vec{m})}$ of stable bordered Riemann surfaces of type (g,h) with $(n,\vec{m})$ marked points (see Definition 3.9, Definition 3.10). There is a finite to one map $\overline{M}_{(g,h),(n,\vec{m})} \to \overline{M}^{\mathbb{R}}_{\tilde{g},(n,m)}$ given by complex double, so there is a finite to one map $\overline{M}_{(g,h),(n,\vec{m})} \to \bar{P}^{\mathbb{R}}_{(\tilde{g},h,0),\tilde{n}} \subset \bar{P}^{\mathbb{R}}_{\tilde{g},\tilde{n}}$. We will first study $\bar{P}_{\tilde{g},\tilde{n}}$ and $\bar{P}^{\mathbb{R}}_{\tilde{g},\tilde{n}}$, following [1, 36].

4.2. **Decomposition into pairs of pants.** A *pair of pants P* is a sphere from which three disjoint closed discs (or points) have been removed. It is the interior a stable bordered Riemann surface of type $(0,3)$. There is a unique hyperbolic structure compatible with the complex structure of P such that the boundary curves are geodesics. Conversely, given a hyperbolic structure on P such that the boundary curves are geodesics, the conformal structure is determined up to conformal or anti-conformal equivalence by the lengths l_1, l_2, and l_3 of the three boundary curves ([1, Chapter II (3.1), Theorem]).

4.2.1. *Riemann surfaces with punctures.* A Riemann surface S of genus $\tilde{g}$ with $\tilde{n}$ punctures can be decomposed into pairs of pants. More precisely, there are $3\tilde{g}-3+\tilde{n}$ disjoint curves $\alpha_1, \ldots, \alpha_{3\tilde{g}-3+\tilde{n}}$ on Σ, each of which is either a closed geodesic (in the hyperbolic metric) or a node, such that the complement of $\cup_{i=1}^{3\tilde{g}-3+\tilde{n}} \alpha_i$ is a disjoint union of $2\tilde{g}-2+\tilde{n}$ pairs of pants $P_1, \ldots, P_{2\tilde{g}-2+\tilde{n}}$. A boundary component of the closure of a pair of pants in this decomposition is either a decomposing curve or a puncture. We call

$$\mathcal{P} = \{P_1, P_2, \ldots, P_{2\tilde{g}-2+\tilde{n}}\}$$

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a *geodesic decomposition of S into pairs of pants.*

Suppose that there exists an anti-holomorphic involution $\sigma : S \to S$. Then (S, σ) is a stable symmetric Riemann surface of genus $\tilde{g}$ with $\tilde{n}$ punctures, and σ is an isometry of the hyperbolic metric. Let

$$\mathcal{P} = \{P_1, P_2, \ldots, P_{2\tilde{g}-2+\tilde{n}}\}$$

be a geodesic decomposition of S into pairs of pants. Then

$$\sigma(\mathcal{P}) = \{\sigma(P_1), \sigma(P_2), \ldots, \sigma(P_{2\tilde{g}-2+\tilde{n}})\}$$

is another geodesic decomposition of S into pairs of pants. $\mathcal{P}$ is said to be *σ-invariant* if $\sigma(\mathcal{P}) = \mathcal{P}$. The argument in [36, Section 4], combined with [1, Chapter II (3.3), Lemma 3], shows that

Theorem 4.1. *Let (S, σ) be a stable symmetric Riemann surface of genus $\tilde{g}$ with $\tilde{n}$ punctures. There exists a σ-invariant geodesic decomposition of Σ into pairs of pants such that the decomposing curves are simple closed geodesics of length less than $C(\tilde{g}, \tilde{n})$, where $C(\tilde{g}, \tilde{n})$ is a constant depending only on $\tilde{g}$, $\tilde{n}$.*

4.2.2. *Riemann surfaces with boundary and punctures.* Let $(\Sigma, \mathbf{B}; \mathbf{p})$ be a stable bordered Riemann surface of type (g, h) with $(n, \vec{0})$ marked points, and suppose that Σ has no boundary nodes. Let S be the complement of marked points in Σ, then S is a surface of type (g, h, n) in the sense of [1], namely, S is obtained by removing h open discs and n points from a compact (possibly nodal) Riemann surface of genus g, where the discs and points are all disjoint. S is stable in the sense that its automorphism group is finite. Let S' be the complement of nodes in S. Each connected component of S' is a smooth Riemann surface with boundary or punctures. The stability condition on S is equivalent to the statement that each connected component of S' has negative Euler characteristic, so there exists a unique hyperbolic metric on S' in the conformal class determined by the complex structure such that the boundary circles are geodesics.

Let S be a stable surface of type (g, h, n). The S can be decomposed into pairs of pants. More precisely, there are $3g + h - 3 + n$ disjoint curves $\alpha_1, \ldots, \alpha_{3g+h-3+n}$ on S, each of which is either a closed geodesic (in the hyperbolic metric) or a node, such that the complement of $\cup_{i=1}^{3g+h-3+n} \alpha_i$ is a disjoint union of $2g + h - 2 + n$ pairs of pants $P_1, \ldots, P_{2g+h-2+n}$. A boundary component of the closure of a pairs of pants in this decomposition is a decomposing curve, a boundary component or a puncture. We call

$$\mathcal{P} = \{P_1, P_2, \ldots, P_{2g+h-2+n}\}$$

a *geodesic decomposition of S into pairs of pants.* We have the following result ([1, Chapter II (3.3), Lemma 3]):

Theorem 4.2. *Let S be a stable surface of type (g, h, n). There is a geodesic decomposition of S into pairs of pants such that the decomposition curves are simple closed geodesic with length less than $C(g, h, n, L_1, \ldots, L_h)$, where $C(g, h, n, L_1, \ldots, L_h)$ is a constant depending only on g, h, n, and the lengths of the h border curves $L_1, \ldots, L_h$.*

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We will see later that the moduli of stable surfaces of type (g,h,n) is of (real) dimension $6g+3h-6+2n$, and $L_1,\dots,L_h$ are among the $6g+3h-6+2n$ real parameters. These are not good coordinates for compactness since $L_1,\dots,L_h$ can be arbitrarily large. Actually, the length of some border curve tends to infinity as S acquires a type H boundary node. To deal with boundary nodes and boundary marked points, we go to the complex double, where the local coordinates of the moduli can be chosen to be bounded.

4.3. **Fenchel-Nielsen coordinates.**

Definition 4.3. *Let S be a stable symmetric Riemann surface of genus $\tilde{g}$ with $\tilde{n}$ punctures. A geodesic pants decomposition $\mathcal{P}$ is* oriented *if*

(1) *The pairs of pants in $\mathcal{P}$ is ordered.*
(2) *The boundary components of each pair of pants in $\mathcal{P}$ is ordered.*
(3) *Any decomposing curve which is not a node is oriented.*

Remark 4.4. *The orientability of a geodesic decomposition of a surface of type (g,h,n) can be defined similarly, with the additional assumption that the boundary components are ordered.*

Let P be a pair of pants with hyperbolic structure with ordered boundary components α_1, α_2, α_3. The base points ξ_i on α_i, $i=1,2,3$, are shown in Figure 16. In Figure 16, $\gamma_{1,2}$ is the geodesic which

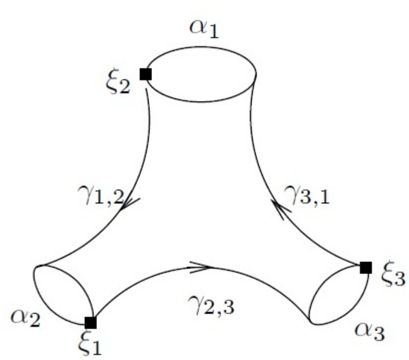

FIGURE 16. a pair of pants

realizes the distance between α_1, α_2, etc.

If $\mathcal{P}$ is oriented, then each boundary curve of a pair of pants in $\mathcal{P}$ has a base point. Each decomposing curve has two distinguished points since it appears twice as a boundary of some pair of pants in $\mathcal{P}$. The orderings in (1) and (2) of Definition 4.3 determine an ordering of the decomposing curves α_j, $j=1,\dots,3\tilde{g}-3+\tilde{n}$ and an ordering on the two distinguished points ξ_j^1,ξ_j^2 on each decomposing curve. We define $l_j(\mathcal{P})$ to be the length of α_j, and $\tau_j(\mathcal{P})$ be the distance one travels from ξ_j^1 to ξ_j^2 along α_j, in the direction determined by part (3) in Definition 4.3. Set $\theta_j(\mathcal{P})=2\pi\frac{\tau_j}{l_j}$ if $l_j\neq 0$. We have $l_j\geq 0$, $0\leq\tau_j<l_j$, and $0\leq\theta_j<2\pi$.

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Definition 4.5. *Let $\tilde{S}$, S be two stable Riemann surfaces of genus $\tilde{g}$ with $\tilde{n}$ punctures. Let $\tilde{S}'$, S' be 1-dimensional complex manifolds obtained from $\tilde{S}$, S by removing the nodes. A* strong deformation *$\kappa : \tilde{S} \to S$ is a continuous map such that*

(1) *If $\tilde{r}$ is a node on $\tilde{S}$, then $\kappa(\tilde{r})$ is a node on S.*
(2) *If r is a node on S, then $\kappa^{-1}(r)$ is a node or an embedded circle on a connected component of S'.*
(3) *$\kappa|_{\kappa^{-1}(S')} : \kappa^{-1}(S') \to S'$ is a diffeomorphism.*

There is a strong deformation $\kappa : \tilde{S} \to S$ if and only if $\tilde{S}$ can be obtained by deforming S as a quasiprojective variety over $\mathbb{C}$.

Remark 4.6. *A strong deformation $\kappa : (\tilde{S}, \tilde{\sigma}) \to (S, \sigma)$ between stable symmetric Riemann surfaces can be defined similarly, with the additional assumption that $\sigma \circ \kappa = \kappa \circ \tilde{\sigma}$. A strong deformation between two surfaces of type (g, h, n) can also be defined similarly.*

We now describe *Fenchel-Nielsen coordinates* for various categories of surfaces.

(1) Let S be a stable Riemann surface of genus $\tilde{g}$ with $\tilde{n}$ punctures. Let $\mathcal{P}$ be an oriented geodesic decomposition of S into pairs of pants, and let $\alpha_1, \alpha_2, \ldots, \alpha_{3\tilde{g}-3+\tilde{n}}$ be the decomposing curves of $\mathcal{P}$. Suppose that there is a strong deformation $\kappa : \tilde{S} \to S$. Let $\tilde{\alpha}_j$ be the closed geodesic homotopic to $\kappa^{-1}(\alpha_j)$. There exists another strong deformation κ' such that $\kappa'(\tilde{\alpha}_j) = \alpha_j$, so $\mathcal{P}$ is pulled back under κ' to an oriented geodesic decomposition $\mathcal{P}_S$.

$$[S] \mapsto (l_1(\mathcal{P}_S), \theta_1(\mathcal{P}_S), \ldots, l_{3\tilde{g}-3+\tilde{n}}(\mathcal{P}_S), \theta_{3\tilde{g}-3+\tilde{n}}(\mathcal{P}_S))$$

defines local coordinates $(l_1, \theta_1, \ldots, l_{3\tilde{g}-3+\tilde{n}}, \theta_{3\tilde{g}-3+\tilde{n}})$ on $\bar{P}_{\tilde{g},\tilde{n}}$. Therefore, both $\overline{M}_{\tilde{g},\tilde{n}}$ and $\bar{P}_{\tilde{g},\tilde{n}}$ are $(6\tilde{g} - 6 + 2\tilde{n})$ dimensional.

(2) Let (S, σ) be a stable symmetric surface of genus $\tilde{g}$ with $\tilde{n}$ punctures, and let $\mathcal{P}$ be an oriented geodesic decomposition of S into pairs of pants which is invariant under σ. By considering σ-invariant decompositions into pants in a neighborhood of (S, σ) in $\bar{P}^{\mathbb{R}}_{\tilde{g},\tilde{n}}$ we obtain local coordinates $(l_1, \theta_1, \ldots, l_{3\tilde{g}-3+\tilde{n}}, \theta_{3\tilde{g}-3+\tilde{n}})$. However, these parameters are not independent. If $\sigma(\alpha_i) = \alpha_j$, $i \neq j$, then $l_i = l_j$, and $\theta_i = c \pm \theta_j$ for some constant c. If $\sigma(\alpha_i) = \alpha_i$, then $\theta_i = 0$. Hence there are $3\tilde{g} - 3 + \tilde{n}$ independent parameters, and the dimension of $\bar{P}^{\mathbb{R}}_{\tilde{g},\tilde{n}}$ is $3\tilde{g} - 3 + \tilde{n}$.

(3) Let S be a stable surface of type (g, h, n), and let $R_1, \ldots, R_h$ be its border curves. Let $\mathcal{P}$ be an oriented geodesic decomposition of S into pairs of pants, and let $\alpha_1, \ldots, \alpha_{3g+h-3+n}$ be the decomposing curves. We have local coordinates $(l_1, \theta_1, \ldots, l_{3g+h-3+n}, \theta_{3g+h-3+n}, L_1, \ldots, L_h)$, where l_j is the length of α_j, θ_j is the angle of gluing along α_j, and L_i is the length of the border curve R_i. Therefore, the dimension of the moduli of stable surfaces of type (g, h, n) is $6g + 3h - 6 + 2n = 3\tilde{g} - 3 + \tilde{n}$, where $\tilde{g} = 2g + h - 1$, and $\tilde{n} = 2n$. This is consistent with the previous paragraph since the complex double of S is a stable symmetric Riemann surface with genus $\tilde{g} = 2g + h - 1$ and $\tilde{n} = 2n$ punctures.

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(4) Let $(\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h)$ be a stable marked bordered Riemann surface of type (g, h) with $(n, \vec{m})$ marked points. $(\Sigma_{\mathbb{C}}, \sigma, \mathbf{x})$ be its complex double, and (S, σ) be the symmetric Riemann surface obtained from Σ by removing the marked points, as in Section 4.1. Choose a σ-invariant geodesic decomposition of S into pants, we have local coordinates $(l_1, \theta_1, \ldots, l_{3\tilde{g}-3+\tilde{n}}, \theta_{3\tilde{g}-3+\tilde{n}})$, as described in (1), where

$$\tilde{g} = 2g + h - 1, \quad \tilde{n} = 2n + m^1 + \cdots + m^h,$$
$$3\tilde{g} - 3 + \tilde{n} = 6g + 3h - 3 + 2n + m^1 + \cdots + m^h.$$

We have seen that half of the $2(6g+3h-6+2n+m^1+\cdots+m^h)$ parameters are independent, so the dimension of $\overline{M}_{(g,h),(n,\vec{m})}$ is

$$6g + 3h - 6 + 2n + m^1 + \cdots + m^h.$$

In the following example, we describe the Fenchel-Nielsen coordinates of the moduli space $\overline{M}_{0,3}$ of a pair of pants explicitly.

Example 4.7. *The hexagon in Figure 17 is obtained by cutting the pair of pants in Figure 16 along the geodesics $\gamma_{1,2}, \gamma_{2,3}, \gamma_{3,1}$. β_1 is the geodesic which realizes the distance between γ_1 and $\gamma_{2,3}$, etc.*

Let $l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8, l_9$ be twice the lengths of $\gamma_1, \gamma_2, \gamma_3, \gamma_{2,3}, \gamma_{3,1}, \gamma_{1,2}, \beta_1, \beta_2, \beta_3$, respectively. The degeneration $l_i = 0$ corresponds to a real codimension one stratum V_i of $\overline{M}_{0,3}$. Let $V_{ij} = V_i \cap V_j$ and $V_{ijk} = V_i \cap V_j \cap V_k$. $\bar{M}_{0,3}$ can be identified with the associahedron K_5 defined by J. Stasheff [40]. The configuration of the strata in $\overline{M}_{0,3} \cong K_5$ is shown in Figure 17. There is 1 three-dimensional stratum. There are 9 two-dimensional strata:

$$V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8, V_9.$$

There are 21 one-dimensional strata:

$$V_{12}, V_{13}, V_{23}, V_{45}, V_{56}, V_{46}, V_{14}, V_{25}, V_{36}, V_{27}, V_{37}, V_{57}, V_{67}, V_{18}, V_{38}, V_{48}, V_{68}, V_{19}, V_{29}, V_{49}, V_{59}.$$

There are 14 zero-dimensional strata:

$$V_{123}, V_{456}, V_{237}, V_{257}, V_{367}, V_{567}, V_{138}, V_{148}, V_{368}, V_{468}, V_{129}, V_{149}, V_{259}, V_{459}.$$

There is a one-to-one correspondence between the 0-dimensional strata and Fenchel-Neilsen coordinate charts of $\overline{M}_{0,3}$: the Fenchel-Neilsen coordinates near V_{ijk} are l_i, l_j, l_k.

4.4. **Compactness and Hausdorffness.** We first define a topology on $\overline{M}_{(g,h),(n,\vec{m})}$, following [1, Chapter II (3.4)], and [36, Section 5]. We will call it the *Fenchel-Nielsen topology.*

Definition 4.8. *A* strong deformation *between two stable marked bordered Riemann surfaces $(\tilde{\Sigma}, \tilde{\mathbf{B}}; \tilde{\mathbf{p}}; \tilde{\mathbf{q}}^1, \ldots, \tilde{\mathbf{q}}^h)$ and $(\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h)$ of type (g, h) with $(n, \vec{m})$ marked points is a continuous map $\kappa : \tilde{\Sigma} \to \Sigma$ such that*

(1) $\kappa(\tilde{B}^i) = B^i$, $\kappa(\tilde{q}^i_k) = q^i_k$, $\kappa(\tilde{p}_j) = p_j$.
(2) *If $\tilde{r}$ is an interior node on $\tilde{\Sigma}$, then $\kappa(\tilde{r})$ is an interior node on Σ.*

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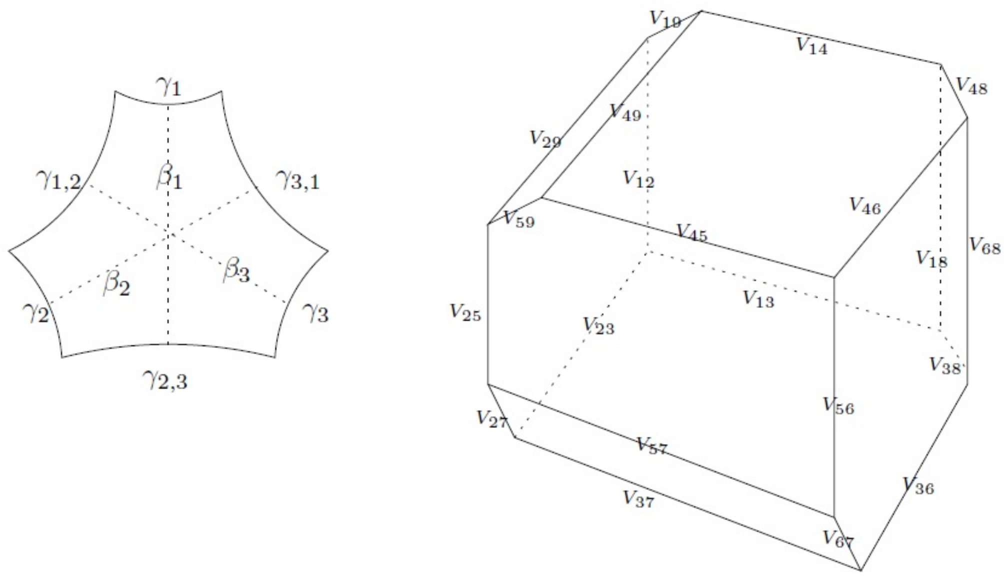

Figure 17.

(3) *If $\tilde{s}$ is a boundary node of type E (H) on $\tilde{\Sigma}$, then $\kappa(\tilde{s})$ is a boundary node of type E (H) on Σ.*

(4) *If r is an interior node on Σ, then $\kappa^{-1}(r)$ is an interior node or a circle.*

(5) *If s is a boundary node of type E, then $\kappa^{-1}(r)$ is a boundary node of type E or a border circle.*

(6) *If s is a boundary node of type H, then $\kappa^{-1}(s)$ is a boundary node or type H or an arc with ends in $\partial\Sigma$.*

(7) *$\kappa|_{\kappa^{-1}(S')} : \kappa^{-1}(S') \to S'$ is a diffeomorphism.*

Let

$$\kappa : (\tilde{\Sigma}, \tilde{\mathbf{B}}; \tilde{\mathbf{p}}; \tilde{\mathbf{q}}^1, \ldots, \tilde{\mathbf{q}}^h) \to (\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h)$$

be a strong deformation. Let $(\tilde{S}, \sigma)$, (S, σ) be the symmetric Riemann surfaces obtained by removing marked points from $\tilde{\Sigma}_{\mathbb{C}}$, $\Sigma_{\mathbb{C}}$, respectively. Define $\kappa_{\mathbb{C}} : \tilde{\Sigma}_{\mathbb{C}} \to \Sigma_{\mathbb{C}}$ by

$$\kappa_{\mathbb{C}}(z) = \begin{cases} \kappa(z) & \text{if } z \in \Sigma \\ \tilde{\sigma} \circ \kappa \circ \sigma(z) & \text{if } z \in \bar{\Sigma} \end{cases}$$

Let $\hat{\kappa}$ denote the restriction of $\kappa_{\mathbb{C}}$ to $\tilde{S}$. Then $\hat{\kappa} : (\tilde{S}, \tilde{\sigma}) \to (S, \sigma)$ is a strong deformation.

Given $\epsilon, \delta > 0$ and

$$\rho = [(\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h)] \in \overline{M}_{(g,h),(n,\vec{m})},$$

we will define a neighborhood $U(\epsilon, \delta, \rho)$ of ρ in $\overline{M}_{(g,h),(n,\vec{m})}$. Let (S, σ) be the symmetric Riemann surface obtained by removing marked points from $\Sigma_{\mathbb{C}}$. Let

$$\tilde{\rho} = [(\tilde{\Sigma}, \tilde{\mathbf{B}}; \tilde{\mathbf{p}}; \tilde{\mathbf{q}}^1, \ldots, \tilde{\mathbf{q}}^h)] \in \overline{M}_{(g,h),(n,\vec{m})},$$

and $(\tilde{S}, \tilde{\sigma})$ be the associated symmetric Riemann surface. Then $\tilde{\rho} \in \overline{M}_{(g,h),(n,\vec{m})}$ if

(1) There exists a σ-invariant oriented geodesic decomposition of S into pairs of pants.

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(2) There exists a strong deformation

$$\kappa : (\tilde{\Sigma}, \tilde{\mathbf{B}}; \tilde{\mathbf{p}}; \tilde{\mathbf{q}}^1, \ldots, \tilde{\mathbf{q}}^h) \to (\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h)$$

in the sense of Definition 4.8. So we have a strong deformation $\hat{\kappa} : \tilde{S} \to S$ as above.

(3) Let l_j, θ_j and $\tilde{l}_j, \tilde{\theta}_j$ be the Fenchel-Nielsen coordinates for $\mathcal{P}$ and $\hat{\kappa}^*(\mathcal{P})$, respectively. Set $d = 6g + 3h - 6 + 2n + m$. We have $|l_j - \tilde{l}_j| < \epsilon$ for $j = 1, \ldots, d$, and $|\theta_j - \tilde{\theta}_j| < \delta$ if $l_j > 0$.

$$\{U(\epsilon, \delta, \rho) \mid \epsilon, \delta > 0, \rho \in \overline{M}_{(g,h),(n,\vec{m})}\}$$

form a basis of the *Fenchel-Nielsen topology*.

$U(\epsilon, \delta, \rho)$ can be described more precisely. Set $z_j = l_j e^{i\theta_j}$, then up to permutation and complex conjugation of some z_k we have

$$\sigma(z_1, z_2, \ldots, z_{2d_1-1}, z_{2d_1}, z_{2d_1+1}, \ldots, z_d) = (\bar{z}_2, \bar{z}_1, \ldots, \bar{z}_{2d_1}, \bar{z}_{2d_1-1}, \bar{z}_{2d_1+1}, \ldots, \bar{z}_d),$$

so the fixed locus of σ consists of points of the form

$$(\bar{z}_2, z_2, \ldots, \bar{z}_{2d_1}, z_{2d_1}, x_1, \ldots, x_{d_2}),$$

where $2d_1 + d_2 = d$, $z_2, z_4, \ldots, z_{2d_1} \in \mathbb{C}$, and $x_1, \ldots, x_{d_2} \in \mathbb{R}$. The coordinates take values in the fixed locus of σ, and x_i are nonnegative on $\overline{M}_{(g,h),(n,\vec{m})}$ because negative values correspond to nonorientable surfaces, as we have seen in Section 3.3. We conclude that $U(\epsilon, \delta, \rho)$ is homeomorphic to $\tilde{U}/\Gamma$, where $\tilde{U}$ is an open subset of $\mathbb{C}^{d_1} \times [0, \infty)^{d_2}$, and Γ is the automorphism group of τ. The transition functions between charts are real analytic [46, Appendix], so Fenchel-Nielsen coordinates give $\overline{M}_{(g,h),(n,\vec{m})}$ the structure of an orbifold with corners. The topology determined by the structure of an orbifold with corners coincides with Fenchel-Nielsen topology. Therefore, we may equip $\overline{M}_{(g,h),(n,\vec{m})}$ with a metric which induces Fenchel-Nielsen topology. In particular, the topology is Hausdorff, and compactness is equivalent to sequential compactness. A straightforward generalization of the argument in [36, Section 6] shows that $\overline{M}_{(g,h),(n,\vec{m})}$ is sequentially compact in Fenchel-Nielsen topology. Therefore,

Theorem 4.9. *$\overline{M}_{(g,h),(n,\vec{m})}$ is Hausdorff and compact in the Fenchel-Nielsen topology.*

4.5. **Orientation.** $\overline{M}_{(g,h),(n,\vec{m})}$ is an orbifold with corners, so we may ask if it is orientable as an orbifold. By Stasheff's results in [40], we have

Theorem 4.10. *$\overline{M}_{(0,1),(0,(m))}$ has $(m-1)!$ isomorphic connected components, which correspond to the cyclic ordering of the m boundary marked points. Each connected component of $\overline{M}_{(0,1),(0,(m))}$ is homeomorphic to $\mathbb{R}^{m-3}$.*

Lemma 4.11. *Suppose that $(g, h, n) \neq (0, 1, 0)$, and $m^i > 0$. If $\overline{M}_{(g,h),(n,\vec{m})}$ is orientable, then $\overline{M}_{(g,h),(n,(m^1,\ldots,m^i+1,\ldots,m^h))}$ is orientable.*

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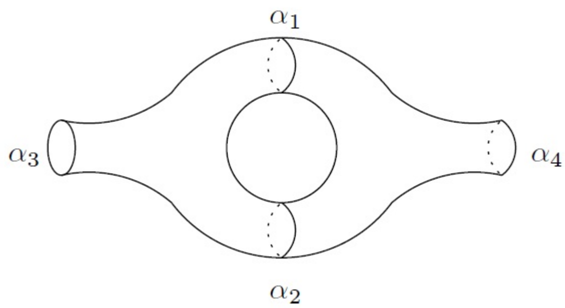

FIGURE 18. $\bar{Q}_{(g,h),n}$ is nonorientable

Proof. Assume that $\overline{M}_{(g,h),(n,(m^1,\ldots,m^i,\ldots,m^h))}$ is orientable. Consider the map

$$F:\overline{M}_{(g,h),(n,(m^1,\ldots,m^i+1,\ldots,m^h))}\to\overline{M}_{(g,h),(n,\vec{m})}, \tag{4.1}$$

given by forgetting the last boundary marked point on the i-th boundary circle. Under our assumption, the fiber of F over $[(\Sigma,\mathbf{B};\mathbf{p};\mathbf{q}^1,\ldots,\mathbf{q}^h)]$ is a union of m^i intervals and inherits the orientation of B^i. Therefore, $\overline{M}_{(g,h),(n,(m^1,\ldots,m^i+1,\ldots,m^h)))}$ is orientable. □

Lemma 4.12. *Suppose that $(g,h,n)\neq(0,1,0)$, and $m^i=0$. If $\overline{M}_{(g,h),(n,(m^1,\ldots,m^i+1,\ldots,m^h)))}$ is orientable, then $\overline{M}_{(g,h),(n,\vec{m})}$ is orientable.*

Proof. Assume that $\overline{M}_{(g,h),(n,(m^1,\ldots,m^i+1,\ldots,m^h)))}$ is orientable. Let T be the tangent bundle of $\overline{M}_{(g,h),(n,\vec{m})}$, which is an orbibundle over $\overline{M}_{(g,h),(n,\vec{m})}$. To show that $\overline{M}_{(g,h),(n,\vec{m})}$ is orientable, it suffices to show that the restriction of T to every loop in $\overline{M}_{(g,h),(n,\vec{m})}$ is orientable. Let $N_{(g,h),(n,\vec{m})}$ be the interior of $\overline{M}_{(g,h),(n,\vec{m})}$. More precisely, $N_{(g,h),(n,\vec{m})}$ corresponds to surfaces with no boundary nodes. Since every loop in $\overline{M}_{(g,h),(n,\vec{m})}$ is homotopic to a loop in $N_{(g,h),(n,\vec{m})}$, it suffices to show that $N_{(g,h),(n,\vec{m})}$ is orientable.

Suppose that $\rho=[(\Sigma,\mathbf{B};\mathbf{p};\mathbf{q}^1,\ldots,\mathbf{q}^h)]\in N_{(g,h),(n,\vec{m})}$. Then B^i is an embedded circle in Σ, oriented as in Remark 2.13, and the fiber of the map F in (4.1) over ρ can be identified with B^i. Thus $N_{(g,h),(n,\vec{m})}$ is orientable. □

It is shown in [20] that

Theorem 4.13. *Suppose that $(g,h,n)\neq(0,1,0)$. Then $\overline{M}_{(g,h),(n,(1,\ldots,1))}$ is a complex orbifold.*

Theorem 4.10, Lemma 4.11, Lemma 4.12, and Theorem 4.13 imply that

Theorem 4.14. *$\overline{M}_{(g,h),(n,\vec{m})}$ is orientable.*

Let $\bar{Q}_{(g,h),n}$ be the moduli space of stable bordered Riemann surfaces of type (g,h) with n interior points. There is an $h!$ to one map $\overline{M}_{(g,h),(n,\vec{0})}\to\bar{Q}_{(g,h),n}$, given by forgetting the ordering of boundary components. Therefore, $\bar{Q}_{(g,h),n}$ is nonorientable.

For example, consider $\rho\in\overline{M}_{(1,2),(0,0)}$ represented by the surface as shown in Figure 18. The local coordinates are $(l_1,\theta_1,l_2,\theta_2,l_3,l_4)$, where l_j is the length of α_j for $j=1,\ldots,4$, and θ_1,θ_2 are gluing

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angles for α_1, α_2, respectively. There is an automorphism ϕ of order 2 of ρ which rotates Figure 18 by 180°. $\phi(\alpha_1) = \alpha_2$, $\phi(\alpha_3) = \alpha_4$, and $\alpha(l_1, \theta_1, l_2, \theta_2, l_3, l_4) = (l_2, -\theta_2, l_1, -\theta_1, l_4, l_3)$, which is orientation reversing.

In general, an automorphism induces permutation of d_1 decomposing curves and permutation of d_2 bordered curves. The former corresponds to permutation of pairs (l_j, θ_j), $j = 1, \ldots, d_1$, which is orientation preserving. The later corresponds to permutation of $(l_{d_1+1}, \ldots, l_{d_1+d_2})$ which is orientation preserving if and only if it is an even permutation. When we consider $\bar{M}_{(g,h),(n,\vec{0})}$, automorphisms permuting border curves are not allowed.

5. Moduli Space of stable maps

5.1. **Prestable and stable maps.** Let (X, ω) be a compact symplectic manifold, and let L be a Lagrangian submanifold. Let J be an ω-tame almost complex structure.

Definition 5.1. *A* prestable map *is a continuous map $u : (\Sigma, \partial\Sigma) \to (X, L)$ such that $J \circ d\hat{u} = d\hat{u} \circ j$, where Σ is a prestable bordered Riemann surface, $\hat{u} = u \circ \tau$, $\tau : \hat{\Sigma} \to \Sigma$ is the normalization map* (*Definition* 3.7).

Definition 5.2. *A* prestable map *of type (g, h) with $(n, \vec{m})$ marked points consists of a prestable marked bordered Riemann surface of type (g, h) with $(n, \vec{m})$ marked points $(\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h)$ and a prestable map $u : (\Sigma, \partial\Sigma) \to (X, L)$.*

Definition 5.3. *A* morphism *between prestable maps of type (g, h) with $(n, \vec{m})$ marked points*

$$(\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h; u) \to (\Sigma', \mathbf{B}'; \mathbf{p}'; (\mathbf{q}')^1, \ldots, (\mathbf{q}')^h; u')$$

is an isomorphism

$$\phi : (\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h) \to (\Sigma', \mathbf{B}'; \mathbf{p}'; (\mathbf{q}')^1, \ldots, (\mathbf{q}')^h)$$

between prestable marked bordered Riemann surfaces of type (g, h) with $(n, \vec{m})$ points such that $u = u' \circ \phi$.

Definition 5.4. *A prestable map of type (g, h) with $(n, \vec{m})$ marked points is* stable *if its automorphism group is finite.*

5.2. C^∞ **topology.** Let $\beta \in H_2(X, L; \mathbb{Z})$, $\vec{\gamma} = (\gamma^1, \ldots, \gamma^h) \in H_1(L; \mathbb{Z})^{\oplus h}$ be such that $\gamma_1 + \cdots + \gamma_h = \partial\beta$, where $\partial : H_2(X, L; \mathbb{Z}) \to H_1(L; \mathbb{Z})$ is the connecting map in the long exact sequence for relative homology groups. Let h be a positive integer, g, n be nonnegative integers, $\vec{m} = (m^1, \ldots, m^h)$ be an h-uple of nonnegative integers, and μ be an integer. Given the above data, define

$$\overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$$

to be the moduli space of isomorphism classes of stable maps of type (g, h) with $(n, \vec{m})$ marked points

$$(\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h; u)$$

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such that $u_*[\Sigma] = \beta$, $u_*[B^i] = \gamma^i$ for $i = 1, \ldots, h$, and $\mu(u^*TX, u^*TL) = \mu$. Here $\mu(u^*TX, u^*TL)$ is the *Maslov index* defined in [24, Definition 3.3.7, Definition 3.7.2]. From now on, we assume that L is oriented, so $\mu(u^*TX, u^*TL)$ is even, and we may restrict ourselves to even μ. We will also assume that γ^i *is nontrivial when* $m^i = 0$*, so the domain cannot have boundary nodes of type* E*.*

Let $\overline{M}_{(g,h),(n,\vec{m})}(X, L)$ be the moduli space of isomorphism classes of stable maps of type (g, h) with $(n, \vec{m})$ marked points. Then $\overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$ are disjoint subsets of $\overline{M}_{(g,h),(n,\vec{m})}(X, L)$ for different $(\beta, \vec{\gamma}, \mu)$.

Let Σ be a prestable bordered Riemann surface, and let $\tau : \hat{\Sigma} \to \Sigma$ be the normalization. Let $u : (\Sigma, \partial\Sigma) \to (X, L)$ be a continuous map such that $\hat{u} = u \circ \tau : (\hat{\Sigma}, \partial\hat{\Sigma}) \to (X, L)$ is C^∞ w.r.t. g_0 on X and some Hermitian metric h on $\hat{\Sigma}$, $l \geq 1$. Define

$$a(u) = \int_{\hat{\Sigma}} \sqrt{\left|\frac{\partial \hat{u}}{\partial x}\right|^2 \left|\frac{\partial \hat{u}}{\partial y}\right|^2 - \left\langle \frac{\partial \hat{u}}{\partial x}, \frac{\partial \hat{u}}{\partial y} \right\rangle^2}\, g(x, y) dx \wedge dy,$$

where (x, y) are local isothermal coordinates on $\hat{\Sigma}$, and $g(x, y)dx \wedge dy$ is the volume form for the metric h. If u is an embedding, $a(u)$ is the area of $u(\Sigma)$ w.r.t. g_0. If u is a prestable map, then $a(u) = \frac{1}{2} \parallel du \parallel_{L^2} = (u_*[\Sigma]) \cap [\omega]$, where

$$\parallel du \parallel_{L^2}^2 = \int_{\hat{\Sigma}} \left(\left|\frac{\partial \hat{u}}{\partial x}\right|^2 + \left|\frac{\partial \hat{u}}{\partial y}\right|^2 \right) g(x, y) dx \wedge dy.$$

$(u_*[\Sigma]) \cap [\omega]$ only depends on the relative homology class $u_*[\Sigma] \in H_2(X, L; \mathbb{Z})$, so we have a function $a : \overline{M}_{(g,h),(n,\vec{m})}(X, L) \to [0, \infty)$, which takes the constant value $\beta \cap [\omega]$ on $\overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$.

With the above definition, $\overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$ is a set. We will equip it with the structure of a topological space, and show that it is sequentially compact and Hausdorff in this topology. This topology was introduced by Gromov [15].

We say two stable maps are close if the complex structures on the domain are close, and the maps are close. To measure the closeness, we use metrics on the domain and on the target. For the target X, J is an ω-tame complex structure, so $g_0(X, Y) = \frac{1}{2}(\omega(X, JY) + \omega(Y, JX))$ is a Riemannian metric on X such that J is an isometry. For the domain, by a *Hermitian metric* h on a prestable bordered Riemann surface Σ we mean a Hermitian metric on $\hat{h}$ on $\hat{\Sigma}$, the normalization of Σ.

We now introduce some notation. Let $\tau : \hat{\Sigma} \to \Sigma$ be the normalization map. Given a node $r \in \Sigma$ and a small positive number ϵ, let $B_\epsilon(r) = \tau(B_\epsilon(r_1) \cup B_\epsilon(r_2))$, where $\tau^{-1}(r) = \{r_1, r_2\}$, and $B_\epsilon(r_\alpha)$ is the geodesic ball of radius ϵ for $\alpha = 1, 2$. Let ϵ be sufficiently small so that $B_\epsilon(r)$ are disjoint for $r \in \Sigma_{\mathrm{sing}}$, where Σ_{sing} denotes the set of nodes on Σ. Set $N_\epsilon(\Sigma) = \bigcup_{r \in \Sigma_{\mathrm{sing}}} B_\epsilon(r)$, $K_\epsilon(\Sigma) = \Sigma - N_\epsilon(\Sigma)$.

Definition 5.5 (C^∞ topology)**.** *Let* $\rho = (\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h; u)$ *be a prestable map of type* (g, h) *with* $(n, \vec{m})$ *marked points. For a Hermitian metric* h *on* Σ *and* $\epsilon_1, \ldots, \epsilon_4 > 0$*, a neighborhood* $U(\rho, h, \epsilon_1, \ldots, \epsilon_4)$ *of* u *in* $\overline{M}_{(g,h),(n,\vec{m})}(X, L)$ *in the* C^∞ topology *is defined as follows. A prestable map*

$$\rho = (\Sigma', \mathbf{B}'; \mathbf{p}'; (\mathbf{q}')^1, \ldots, (\mathbf{q}')^h; u')$$

belongs to $U(\rho, h, \epsilon_1, \ldots, \epsilon_4)$ *if*

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(1) *There is a strong deformation*

$$\kappa : (\Sigma', \mathbf{B}'; \mathbf{p}'; (\mathbf{q}')^1, \ldots, (\mathbf{q}')^h) \to (\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h)$$

such that κ^{-1} *is defined on* $K_{\epsilon_1}(\Sigma)$.

(2) $\| j - (\kappa^{-1})^* j' \|_{C^\infty(K_{\epsilon_1}(\Sigma))} < \epsilon_2$, *where* j, j' *are complex structures on* Σ, Σ', *respectively.*

(3) $\| u - u' \circ \kappa^{-1} \|_{C^\infty(K_{\epsilon_1}(\Sigma))} < \epsilon_3$.

(4) $|a(u) - a(u')| < \epsilon_4$.

(1) says that Σ' can be obtained by deforming Σ, or equivalently, Σ' is in the same or a higher stratum in $\widetilde{M}_{(g,h),(n,\vec{m})}$, the moduli space of *prestable* marked bordered Riemann surfaces of type (g, h) with $(n, \vec{m})$ marked points. (2) says that

$$(\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h),\ (\Sigma', \mathbf{B}'; \mathbf{p}'; (\mathbf{q}')^1, \ldots, (\mathbf{q}')^h)$$

are close in the C^∞ topology (Definition 5.6). (3) says that the maps u, u' are C^∞ close away from the nodes. Finally (4) implies that $a : \overline{M}_{(g,h),(n,\vec{m})}(X, L) \to [0, \infty)$ is a continuous function.

$$(\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h; u), \quad (\Sigma', \mathbf{B}'; \mathbf{p}'; (\mathbf{q}')^1, \ldots, (\mathbf{q}')^h; u')$$

represent the same point in $\overline{M}_{(g,h),(n,\vec{m})}(X, L)$ if and only if there is a Hermitian metric h on Σ such that

(1) There is a homeomorphism

$$\kappa : (\Sigma', \mathbf{B}'; \mathbf{p}'; (\mathbf{q}')^1, \ldots, (\mathbf{q}')^h) \to (\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h)$$

which induces a diffeomorphism $\hat{\Sigma}' \to \hat{\Sigma}$.

(2) $\| j - (\kappa^{-1})^* j' \|_{C^\infty(\Sigma)} = 0$, where j, j' are complex structures on Σ, Σ', respectively.

(3) $\| u - u' \circ \kappa^{-1} \|_{C^\infty(\Sigma)} = 0$.

(4) $|a(u) - a(u')| = 0$.

This shows that the C^∞ topology is actually a topology on the moduli space $\overline{M}_{(g,h),(n,\vec{m})}(X, L)$. $\overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$ is a closed subspace of $\overline{M}_{(g,h),(n,\vec{m})}(X, L)$ and is equipped with the subspace topology.

When X is a point, we get the C^∞ topology of $\overline{M}_{(g,h),(n,\vec{m})}$.

Definition 5.6. *Let* $\lambda = (\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h)$ *be a stable (prestable) bordered Riemann surface of type* (g, h) *with* $(n, \vec{m})$ *marked points. For a Hermitian metric* h *on* Σ *and* $\epsilon_1, \epsilon_2 > 0$, *a neighborhood* $U(\rho, h, \epsilon_1, \epsilon_2)$ *of* u *in* $\overline{M}_{(g,h),(n,\vec{m})}$ *in the* C^∞ topology *is defined as follows. A stable (prestable) bordered Riemann surface* $\lambda' = (\Sigma', \mathbf{B}'; \mathbf{p}'; (\mathbf{q}')^1, \ldots, (\mathbf{q}')^h)$ *belongs to* $U(\rho, h, \epsilon_1, \epsilon_2)$ *if*

(1) *There is a strong deformation* $\kappa : \lambda' \to \lambda$ *such that* κ^{-1} *is defined on* $K_{\epsilon_1}(\Sigma)$.

(2) $\| j - (\kappa^{-1})^* j' \|_{C^\infty(K_{\epsilon_1}(\Sigma))} < \epsilon_2$, *where* j, j' *are complex structures on* Σ, Σ', *respectively.*

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5.3. **Compactness and Hausdorffness.** The following result is the main theorem of this section.

Theorem 5.7. *$\overline{M}_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)$ is Hausdorff and sequentially compact in the C^∞ topology.*

The Hausdorffness can be proven as in the case of curves without boundary, see e.g. [37, Proposition 3.8]. The compactness is a consequence of the following theorem.

Theorem 5.8 (Gromov Compactness Theorem)**.** *Let $\{\rho_l\}$ be a sequence in $\overline{M}_{(g,h),(n,\vec{m})}(X,L)$ such that $a(\rho_l)<C$ for all $l\in\mathbb{N}$. Then there is a subsequence of $\{\rho_l\}$ convergent in the C^∞ topology.*

The proof of Gromov compactness theorem [15, 1.5] for J-holomorphic curves without boundary was carried out in details in [34, 47]. The case with boundary was proved in [47] (see also [19, 20]). In [47], the moduli space is compactified by the moduli space of *cusp curves*, or prestable maps of this paper. We will describe how the argument in [47] proves Theorem 5.8.

The C^∞ topology can be equivalently defined as follows.

Definition 5.9 (C^∞ Topology)**.** *A sequence*

$$\rho_l=(\Sigma_l,\mathbf{B}_l;\mathbf{p}_l;\mathbf{q}_l^1,\ldots,\mathbf{q}_l^h;u_l)$$

converges to $\rho=(\Sigma,\mathbf{B};\mathbf{p};\mathbf{q}^1,\ldots,\mathbf{q}^h;u)$ in the C^∞ topology if for each $\epsilon_1,\ldots,\epsilon_4>0$, there is an integer N such that for $l\geq N$,

(1) *There is a strong deformation $\kappa_l:\Sigma_l\to\Sigma$ such that κ_l^{-1} is defined on $K_{\epsilon_1}(\Sigma)$.*
(2) $\parallel j-(\kappa_l^{-1})^*j_l\parallel_{C^\infty(K_{\epsilon_1}(\Sigma))}<\epsilon_2$.
(3) $\parallel u-u_l\circ\kappa_l^{-1}\parallel_{C^\infty(K_{\epsilon_1}(\Sigma))}<\epsilon_3$.
(4) $|a(u)-a(u_l)|<\epsilon_4$.

Recall that $\widetilde{M}_{(g,h),(n,\vec{m})}$ denotes the moduli space of prestable bordered Riemann surfaces of type (g,h) with $(n,\vec{m})$ marked points. There is a map $F:\overline{M}_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)\to\widetilde{M}_{(g,h),(n,\vec{m})}$, given by forgetting the map. $\widetilde{M}_{(g,h),(n,m)}$ has infinitely many strata since one can keep on going to lower and lower strata by adding non-stable components – spheres and discs.

We claim that the image of F is covered by only finitely many strata, or equivalently,

Lemma 5.10. *The domains in $\overline{M}_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)$ have only finitely many topological types.*

Proof. There is a map $\widetilde{M}_{(g,h),(n,\vec{m})}\to\overline{M}_{(g,h),(n,\vec{m})}$, given by contracting non-stable components. Since a stable bordered Riemann surfaces of type (g,h) with $(n,\vec{m})$ points can have only finitely many possible topological types, it suffices to get an upper bound for the number of non-stable irreducible components. The restriction of a stable map to a non-stable irreducible component is nonconstant, so there is a lower bound $\epsilon>0$ for the area of the restriction of the map to each non-stable component by [47, Lemma 4.3, Lemma 4.5]. Therefore, the number of non-stable irreducible components cannot exceed $(\beta\cap[\omega])/\epsilon$. □

Let $\{\rho_l\}$ be a sequence in $\bar{M}_{(g,h),(n,\vec{m})}(X,L)$ such that $a(\rho)<C$ for all $l\in\mathbb{N}$. By Lemma 5.10, there is a subsequence of $\{\rho_l\}$ such that the domains are of the same topological type. By normalization

we obtain several sequences of stable maps with smooth domains of the same topological type and with uniform area bound. Note that each node gives rise to two marked points on the normalization. It suffices to show that each sequence has a subsequence convergent in the C^∞ topology. Therefore, we may assume that the domain is a smooth marked bordered Riemann surface or a smooth curve with marked points. In this case, it is proven in [47] that there is a subsequence which converges to a prestable map in the C^∞ topology. However, it is straightforward to check that the limit produced in [47] is actually a stable map.

6. Construction of Kuranishi structure

6.1. **Kuranishi structure with corners.** We first quote the following definition from [10, A2.1.1-A2.1.4], which is a slight modification of [9, Definition 5.1].

Definition 6.1 (Kuranishi neighborhood)**.** *Let M be a Hausdorff topological space. A* Kuranishi neighborhood (with corners) *of $p \in M$ is a 5-uple $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ such that*

(1) *V_p is a smooth manifold (with corners), and E_p is a smooth vector bundle on it.*
(2) *Γ_p is a finite group which acts smoothly on $E_p \to V_p$.*
(3) *s_p is a Γ_p-equivariant continuous section of E_p.*
(4) *$\psi_p : s_p^{-1}(0) \to M$ is a continuous map which induces a homeomorphism from $s_p^{-1}(0)/\Gamma_p$ to a neighborhood of p in M.*

We call E_p the obstruction bundle *and s_p the* Kuranishi map*.*

The following equivalence relation is weaker than the one in [9, Definition 5.2], so the resulting equivalence class is larger.

Definition 6.2. *Let M be a Hausdorff topological space. Two Kuranishi neighborhoods (with corners) $(V_{1,p}, E_{1,p}, \Gamma_{1,p}, \psi_{1,p}, s_{1,p})$ and $(V_{2,p}, E_{2,p}, \Gamma_{2,p}, \psi_{2,p}, s_{2,p})$ of $p \in M$ are* equivalent *if*

(1) $\dim V_{1,p} - \mathrm{rank} E_{1,p} = \dim V_{2,p} - \mathrm{rank} E_{2,p} \equiv d$.
(2) *There is another Kuranishi neighborhood (with corners) $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ of p such that $\dim V_p - \mathrm{rank} E_p = d$.*
(3) *There are homomorphisms $h_i : \Gamma_{i,p} \to \Gamma_p$ for $i = 1, 2$.*
(4) *For $i = 1, 2$, there is a $\Gamma_{i,p}$-invariant open neighborhood V_i of $\psi_{i,p}^{-1}(p)$, an h_i-equivariant embedding $\phi_i : V_i \to V_p$, and an h_i-equivariant embedding of vector bundles $\hat{\phi}_i : E_{i,p}|_{V_i} \to E_p$ which covers ϕ_i.*
(5) *$\hat{\phi}_i \circ s_{i,p} = s_p \circ \phi_i$ for $i = 1, 2$.*
(6) *$\psi_{i,p} = \psi_p \circ \phi_i$ for $i = 1, 2$.*

In this case, we write $(V_{1,p}, E_{1,p}, \Gamma_{1,p}, \psi_{1,p}, s_{1,p}) \sim (V_{2,p}, E_{2,p}, \Gamma_{2,p}, \psi_{2,p}, s_{2,p})$

The following definition is a combination of [10, A2.1.5-A2.1.11] and [9, Definition 5.3].

Definition 6.3 (Kuranishi structure)**.** *Let M be a Hausdorff topological space. A* Kuranishi structure (with corners) *on M assigns a Kuranishi neighborhood (or a Kuranishi neighborhood with corners) $(V_p,$*

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$E_p, \Gamma_p, \psi_p, s_p)$ to each $p \in M$ and a 4-uple $(V_{pq}, \hat{\phi}_{pq}, \phi_{pq}, h_{pq})$ to each pair (p,q) where $p \in M, q \in \psi_p(s_p^{-1}(0))$ such that

1. *V_{pq} is an open subset of V_q containing $\psi_q^{-1}(q)$.*
2. *h_{pq} is a homomorphism $\Gamma_q \to \Gamma_p$.*
3. *$\phi_{pq} : V_{pq} \to V_p$ is an h_{pq}-equivariant embedding.*
4. *$\hat{\phi}_{pq} : E_q|_{V_{pq}} \to E_p$ is an h_{pq}-equivariant embedding of vector bundles which covers ϕ_{pq}.*
5. *$\hat{\phi}_{pq} \circ s_q = s_p \circ \phi_{pq}$.*
6. *$\psi_q = \psi_p \circ \phi_{pq}$.*
7. *If $r \in \psi_q(s_q^{-1}(0) \cap V_{pq})$, then $\hat{\phi}_{pq} \circ \hat{\phi}_{qr} = \hat{\phi}_{pr}$ in a neighborhood of $\psi_r^{-1}(r)$.*
8. $\dim V_p - \mathrm{rank}\, E_p$ *is independent of p and is called the* virtual dimension *of the Kuranishi structure (with corners).*

$(V_{pq}, \hat{\phi}_{pq}, \phi_{pq}, h_{pq})$ is called a transition function *from $(V_q, E_q, \Gamma_q, \psi_q, s_q)$ to $(V_p, E_p, \Gamma_p, \psi_p, s_p)$.*

Remark 6.4. *Let M be a Hausdorff space with a Kuranishi structure with corners*

$$\mathcal{K} = \Big\{ (V_p, E_p, \Gamma_p, \psi_p, s_p) : p \in M, (V_{pq}, \hat{\phi}_{pq}, \phi_{pq}, h_{pq}) : q \in \psi_p(s_p^{-1}(0)) \Big\}$$

of virtual dimension d. Let $\partial M = \cup_{p \in M} \psi_p(s_p^{-1}(0) \cap \partial V_p)$, where ∂V_p is the union of corners in V_p. Then

$$\partial\mathcal{K} = \Big\{ (\partial V_p, E_p|_{\partial V}, \Gamma_p, \psi_p, s_p) : p \in \partial M, (\partial V_{pq}, \hat{\phi}_{pq}, \phi_{pq}, h_{pq}) : q \in \psi_p(s_p^{-1}(0) \cap \partial V_p) \Big\}$$

is a Kuranishi structure with corners of virtual dimension $d-1$ on ∂M.

Definition 6.5. *Let M be a Hausdorff topological space. Two Kuranishi structures*

$$\mathcal{K}_1 = \Big\{ (V_{1,p}, E_{1,p}, \Gamma_{1,p}, \psi_{1,p}, s_{1,p}) : p \in M, (V_{1,pq}, \hat{\phi}_{1,pq}, \phi_{1,pq}, h_{1,pq}) : q \in \psi_{1,p}(s_{1,p}^{-1}(0)) \Big\}$$

and

$$\mathcal{K}_2 = \Big\{ (V_{2,p}, E_{2,p}, \Gamma_{2,p}, \psi_{2,p}, s_{2,p}) : p \in M, (V_{2,pq}, \hat{\phi}_{2,pq}, \phi_{2,pq}, h_{2,pq}) : q \in \psi_{2,p}(s_{2,p}^{-1}(0)) \Big\}$$

on M are equivalent *if there is another Kuranishi structure*

$$\mathcal{K} = \Big\{ (V_p, E_p, \Gamma_p, \psi_p, s_p) : p \in M, (V_{pq}, \hat{\phi}_{pq}, \phi_{pq}, h_{pq}) : q \in \psi_p(s_p^{-1}(0)) \Big\}$$

on M such that for all $p \in M$, $(V_{1,p}, E_{1,p}, \Gamma_{1,p}, \psi_{1,p}, s_{1,p})$, $(V_{2,p}, E_{2,p}, \Gamma_{2,p}, \psi_{2,p}, s_{2,p})$, and $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ satisfy the relation described in Definition 6.2. In this case, we write $\mathcal{K}_1 \sim \mathcal{K}_2$.

Let $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ be a Kuranishi neighborhood (with corners) of p. If s_p intersects the zero section of E_p transversally, then $\tilde{M}_p = s_p^{-1}(0)$ is a smooth submanifold (with corners) of V_p of dimension $\dim V_p - \mathrm{rank} E_p$, and there is an exact sequence of smooth vector bundles

$$0 \to T\tilde{M}_p \to TV_p|_{\tilde{M}_p} \stackrel{ds_p}{\to} E_p|_{\tilde{M}_p}$$

over $\tilde{M}_p$. In particular, $T\tilde{M}_p$ is equivalent to the two term complex $[TV_p|_{\tilde{M}_p} \stackrel{ds_p}{\to} E_p|_{\tilde{M}_p}]$ as an element of the Grothendieck group $KO(\tilde{M}_p)$.

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Both TV_p and E_p are Γ_p-equivariant vector bundles over V_p, so TV_p/Γ_p, E_p are orbibundles over the orbifold (with corners) $U_p = V_p/\Gamma_p$. We call $TU_p \equiv TV_p/\Gamma_p$ the tangent bundle of the orbifold (with corners) U_p, and $TM_p \equiv T\tilde{M}_p/\Gamma_p$ is the tangent bundle of the orbifold (with corners) $M_p = \tilde{M}_p/\Gamma_p$.

In general, $\tilde{M}_p$ might be singular, so $T\tilde{M}_p$ does not exist. Nevertheless, $\tilde{M}_p$ is a topological space, and $KO(\tilde{M}_p)$ makes sense. We define

$$T^{\mathrm{vir}}\tilde{M}_p = [TV_p|_{\tilde{M}_p} \stackrel{ds_p}{\to} E_p|_{\tilde{M}_p}] \in KO(\tilde{M}_p)$$

to be the *virtual tangent bundle* of $\tilde{M}_p$, and

$$T^{\mathrm{vir}}M_p = [TU_p|_{M_p} \stackrel{ds_p}{\to} (E_p/\Gamma_p)|_{M_p}]$$

to be the *virtual tangent bundle* of M_p. We have $T^{\mathrm{vir}}\tilde{X}_p = T\tilde{M}_p$ and $T^{\mathrm{vir}}M_p = TM_p$ when s_p intersects the zero section transversally. The transition functions $(V_{pq}, \hat{\phi}_{pq}, \phi_{pq}, h_{pq})$ in Definition 6.3 enable us to glue $T^{\mathrm{vir}}M_p$ to obtain the *virtual tangent bundle* $T^{\mathrm{vir}}M$ of the Kuranishi structure on M.

$\det TV_p \otimes (\det E_p)^{-1}$ glue to a real line orbibundle $\det(T^{\mathrm{vir}}M)$, the *orientation bundle* of the Kuranishi structure (with corners). It is a real line bundle if the action of each Γ_p on $\det TV_p \otimes (\det E_p)^{-1}$ is orientation preserving. We say a Kuranishi structure is *orientable* if its orientation bundle is a trivial real line bundle. If $\mathcal{K}$ and $\mathcal{K}'$ are equivalent Kuranishi structures (with corners), then $\mathcal{K}$ is orientable if and only if $\mathcal{K}'$ is orientable.

In the ordinary Gromov-Witten theory, there is an algebraic approach to define Gromov-Witten invariants when the target is a smooth projective variety [7]. In the algebraic approach, the moduli of stable maps is a *Deligne-Mumford stack*, which is locally étale covered by affine schemes. In Definition 6.3, $s_p^{-1}(0)$ is the analogue of an affine scheme – an affine scheme is the zero locus of polynomials, while $s_p^{-1}(0)$ is the zero locus of smooth functions.

The moduli space of stable maps admits a *perfect obstruction theory*, which is an element in the derived category locally isomorphic to a two term complex of vector bundles $[E_{-1} \to E_0]$. Given a perfect obstruction theory, the *virtual dimension* is defined to be $\mathrm{rank}\, E_0 - \mathrm{rank}\, E_{-1}$, and a virtual fundamental class of the virtual dimension can be constructed. The two term complex $[TV_p|_{\tilde{M}_p} \stackrel{ds_p}{\to} E_p|_{\tilde{M}_p}]$ is the analogue of $[E_0^\vee \to E_{-1}^\vee]$.

A Kuranishi structure can be viewed as the analytic counterpart of a Deligne-Mumford stack together with a perfect obstruction theory. We will show that

Theorem 6.6. *$\overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$ has a Kuranishi structure of virtual dimension*

$$\mu + (N-3)(2-2g-h) + 2n + m^1 + \cdots + m^h,$$

where $2N$ is the dimension of X. The Kuranishi structure is orientable if L is spin or if $h = 1$ and L is relatively spin, i.e., L is orientable and $w_2(TL) = \alpha|_L$ for some $\alpha \in H^2(X, \mathbb{Z}_2)$.

6.2. **Stable $W^{k,p}$ maps.** Let (X, ω) be a compact symplectic manifold together with an ω-tame almost complex structure J, and let L be a compact Lagrangian submanifold of X as before. To construct a Kuranishi structure on $\overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$, we need to enlarge the category of stable maps. We first specify metrics on the target (Section 6.2.1) and on the domain (Section 6.2.2) which enables us to define norms on relevant Banach spaces. The definition of stable $W^{k,p}$ maps is given in Section 6.2.3. The virtual dimension in Theorem 6.6 is computed in Section 6.2.4.

6.2.1. *Metric on the target.* Let g_0 be the Riemannian metric on X defined by $g_0(v,w)=\frac{1}{2}(\omega(v,Jw)+\omega(w,Jv))$. We will modify g_0 to obtain a Riemannian metric g_1 such that L is totally geodesic w.r.t. g_1.

Lemma 6.7. *Given a Riemannian vector bundle (V,h) over a compact Riemannian manifold (M,g), there is a Riemannian metric $\tilde{g}$ on the total space of V such that*

1. *For any $x\in M$, the restriction of $\tilde{g}$ to the fiber V_x over x is $h(x)$.*
2. *The zero section $i_0:(M,g)\to(V,\tilde{g})$ is an isometric embedding.*
3. *$i_0(M)$ is totally geodesic in $(V,\tilde{g})$.*

Proof. Let $\pi:V\to M$ be the canonical projection. Choose a connection on V which is compatible with h. This gives a decomposition $TV=\pi^*V\oplus H$, where $H\cong\pi^*TM$. Let $x\in M$, $w\in V_x$, so that $(x,w)\in V$. Given $\xi\in T_{(x,w)}V$, there is a unique decomposition $\xi=\xi_v+\xi_h$, where $\xi_v\in\pi^*V$, and $\xi_h\in H$. Define a quadratic form Q on $T_{(x,w)}V$ by $Q(\xi,\xi)=h(x)(\xi_v,\xi_v)+g(x)(\pi_*(\xi_h),\pi_*(\xi_h))$. Then Q determines an inner product $\tilde{g}(x,w)$ on $T_{(x,w)}V$. $\tilde{g}$ is a Riemannian metric on V which clearly satisfies (1) and (2).

For (3), let $x_0,x_1\in i_0(M)$ be close enough such that there is a unique length minimizing geodesic $\gamma:[0,1]\to(V,\tilde{g})$ such that $\gamma(0)=x_0$, $\gamma(1)=x_1$. It suffices to show that this geodesic lies in $i_0(M)$. We have

$$\begin{aligned}
l(\gamma) &= \int_0^1 \tilde{g}(\gamma',\gamma')dt\\
&= \int_0^1 (h(\pi\circ\gamma)(\gamma_v',\gamma_v')+g(\pi\circ\gamma)((\pi\circ\gamma)',(\pi\circ\gamma)'))dt\\
&\geq \int_0^1 g(\pi\circ\gamma)((\pi\circ\gamma)',(\pi\circ\gamma)'))dt\\
&= \int_0^1 \tilde{g}((i_0\circ\pi\circ\gamma)',(i_0\circ\pi\circ\gamma)')dt\\
&= l(i_0\circ\pi\circ\gamma),
\end{aligned}$$

The equality holds since γ is length minimizing. Hence $\gamma_v'\equiv 0$, and $\gamma=i_0\circ\pi\circ\gamma:[0,1]\to L$. □

The Riemannian metric g_0 on X gives an orthogonal decomposition

$$TX|_L=TL\oplus N_{L/X},$$

where $N_{L/X}$ is the normal bundle of L in X. Let $\exp^0$ denote the exponential map $TX\to X$ determined by g_0. For $R>0$, let $B_R(TX)$ denote the ball bundle of radius R in TX. There exists $R>0$ such that $\exp^0$ maps $B_R(N_{L/X})$ diffeomorphically to its image in X. For $r\leq R$, let $N_r(L)$ denote the image of $B_r(N_{L/X})$ under $\exp^0$. We have a diffeomorphism $G:N_R(L)\to B_R(N_{L/X})$ which is the inverse of

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$\exp^0|_{B_R(N_{L/X})}$. Then we could construct a Riemannian metric $\tilde{g}$ on $N_{L/X}$ as in Lemma 6.7. Let χ be a smooth cut-off function defined on X such that $\chi = 1$ on $N_{\frac{R}{3}}(L)$ and $\chi = 0$ on $X - N_{\frac{2R}{3}}(L)$. Define $g_1 = \chi G^*\tilde{g} + (1-\chi)g_0$ on $N_R(L)$. Then $g_1 = g_0$ on $N_R(L) - N_{\frac{2R}{3}}(L)$, so g_1 extends to a Riemannian metric on X such that $g_1 = g_0$ on $X - N_R(L)$ and on $TX|_L$.

X is compact, so there exists some constant $C_0 > 0$ such that $C_0^{-1} g_0 \leq g_1 \leq C_0 g_0$. The C^∞ topology (Definition 5.5) defined by g_1 is equivalent to that defined by g_0. From now on, all the parallel transports, exponential maps, and norms are defined by g_1 instead of g_0.

6.2.2. *Metric on the domain.* Let $\lambda = [(\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \dots, \mathbf{q}^h)] \in \widetilde{M}_{(g,h),(n,\vec{m})}$, and let $\overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)_\lambda$ denote the fiber of

$$F : \overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu) \to \widetilde{M}_{(g,h),(n,\vec{m})}$$

over λ. Choose a Hermitian metric $\tilde{h}$ on $\Sigma_{\mathbb{C}}$ which is compact, flat near nodes, and invariant under the antiholomorphic involution $\sigma : \Sigma_{\mathbb{C}} \to \Sigma_{\mathbb{C}}$. Let h be the restriction of $\tilde{h}$ to Σ. Then the border curves of Σ are geodesics in the Riemannian metric determined by h. We further require that

(1) If s is an interior node, then there is an isometric holomorphic embedding $B_\epsilon(r) \to \mathbb{C}^2$, where $\mathbb{C}^2$ is equipped with the standard metric, such that the image is $\{(x,y) \in \mathbb{C}^2 \mid xy = 0, |x| < \epsilon, |y| < \epsilon\}$.
(2) If s is a boundary node of type H, then there is an isometric holomorphic embedding $B_\epsilon(r) \to \mathbb{C}^2/A$, where $A(x,y) = (\bar{x}, \bar{y})$, such that the image is $\{(x,y) \in \mathbb{C}^2 \mid xy = 0, |x| < \epsilon, |y| < \epsilon\}/A$.
(3) h is invariant under $\mathrm{Aut}\,\rho$.

We call such a Hermitian metric an *admissible metric*.

6.2.3. *$W^{k,p}$ maps and C^l maps.*

Definition 6.8. *Let Σ be a prestable bordered Riemann surface. A continuous map $u : (\Sigma, \partial\Sigma) \to (X, L)$ is a $W^{k,p}$* map on λ *if $\hat{u} = u \circ \tau : (\hat{\Sigma}, \partial\hat{\Sigma}) \to (X, L)$ is of class $W^{k,p}$ in the sense of* [31, Appendix B]*, where $\tau : \hat{\Sigma} \to \Sigma$ is the normalization map.*

In the above definition, we assume that $kp > 2$, so the embedding $W^{k,p} \subset C^0$ is compact.

Definition 6.9. *A* prestable $W^{k,p}$ map *of type (g,h) with $(n, \vec{m})$ marked points consists of a prestable marked bordered Riemann surface of type (g,h) with $(n,\vec{m})$ marked points $(\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \dots, \mathbf{q}^h)$ and a prestable $W^{k,p}$ map $u : (\Sigma, \partial\Sigma) \to (X, L)$.*

Definition 6.10. *A* morphism *between prestable $W^{k,p}$ maps of type (g,h) with $(n,\vec{m})$ marked points*

$$(\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \dots, \mathbf{q}^h; u) \to (\Sigma', \mathbf{B}'; \mathbf{p}'; (\mathbf{q}')^1, \dots, (\mathbf{q}')^h; u')$$

is an isomorphism

$$\phi : (\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \dots, \mathbf{q}^h) \to (\Sigma', \mathbf{B}'; \mathbf{p}'; (\mathbf{q}')^1, \dots, (\mathbf{q}')^h)$$

between prestable bordered Riemann surfaces of type (g,h) with $(n,\vec{m})$ points such that $u = u' \circ \phi$.

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Definition 6.11. *A prestable $W^{k,p}$ map of type (g,h) with $(n,\vec{m})$ marked points is* stable *if its automorphism group is finite.*

C^l maps and stable C^l maps are defined similarly.

Let $W^{k,p}_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu), C^l_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)$ be the moduli space of isomorphism classes of stable $W^{k,p}$, C^l maps of type (g,h) with $(n,\vec{m})$ marked points satisfying the topological conditions as in the definition of $\overline{M}_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)$, respectively. There are maps

$$F^{k,p}: W^{k,p}_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)\to\widetilde{M}_{(g,h),(n,\vec{m})}$$

and

$$C^l: C^l_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)\to\widetilde{M}_{(g,h),(n,\vec{m})}$$

given by forgetting the map. Recall that forgetting the map also gives

$$F:\overline{M}_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)\to\widetilde{M}_{(g,h),(m,n)}.$$

Given $\lambda\in\widetilde{M}_{(g,h),(m,n)}$, let $\overline{M}_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)_\lambda$, $W^{k,p}_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)_\lambda$, $C^l_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)_\lambda$ denote the fiber of F, $F^{k,p}$, F^l over λ, respectively. From now on, we will write

$$\begin{aligned}
M_\lambda &= \overline{M}_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)_\lambda\\
W^{k,p}_\lambda &= W^{k,p}_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)_\lambda\\
C^l_\lambda &= C^l_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)_\lambda
\end{aligned}$$

for convenience.

Let exp denote the exponential map of the Riemannian metric g_1 on X, and let h be an admissible metric on Σ. For a stable $W^{k,p}$ map u on λ and $\epsilon>0$, define

$$U^{k,p}(u,\epsilon)=\{\exp_u(w)\mid w\in W^{k,p}(\Sigma,\partial\Sigma,u^*TX,(u|_{\partial\Sigma})^*TL),\parallel w\parallel_{W^{k,p}}<\epsilon\},$$

where $W^{k,p}(\Sigma,\partial\Sigma,u^*TX,(u|_{\partial\Sigma})^*TL)$ will be defined in Section 6.2.4, and the norms are defined by g_1, h. Similarly, for a stable C^l map u on λ and $\epsilon>0$, define

$$U^l(u,\epsilon)=\{\exp_u(w)\mid w\in C^l(\Sigma,\partial\Sigma,u^*TX,(u|_{\partial\Sigma})^*TL),\parallel w\parallel_{C^l}<\epsilon\},$$

where $C^l(\Sigma,\partial\Sigma,u^*TX,(u|_{\partial\Sigma})^*TL)$ will be defined in Section 6.2.4. Note that $\exp_u(w)|_{\partial\Sigma}\subset L$ since L is totally geodesic w.r.t. g_1. Then

$$\{U^{k,p}(u,\epsilon)\mid u\text{ is a stable }W^{k,p}\text{ map on }\lambda,\epsilon>0\}$$

generate the $W^{k,p}$ topology on $W^{k,p}_\lambda$, while

$$\{U^l(u,\epsilon)\mid u\text{ is a stable }C^l\text{ map on }\lambda,\epsilon>0\}$$

generate the C^l topology on C^l_λ.

Define

$$W^{k,p}(u,\epsilon)=\{w\in W^{k,p}(\Sigma,\partial\Sigma,u^*TX,(u|_{\partial\Sigma})^*TL)\mid\parallel w\parallel_{W^{k,p}}<\epsilon\}$$

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and

$$C^l(u,\epsilon)=\{w\in C^l(\Sigma,\partial\Sigma,u^*TX,(u|_{\partial\Sigma})^*TL)\mid \parallel w\parallel_{C^l}<\epsilon\}.$$

For sufficiently small $\epsilon>0$, $w\mapsto \exp_u(w)$ gives

$$U^{k,p}(u,\epsilon)\cong W^{k,p}(u,\epsilon)/\mathrm{Aut}(\lambda,u)$$

and

$$U^l(u,\epsilon)\cong C^l(u,\epsilon)/\mathrm{Aut}(\lambda,u).$$

Therefore, $W^{k,p}_\lambda$ and C^l_λ are Banach orbifolds. They are Banach manifolds if $\mathrm{Aut}\lambda$ is trivial. M_λ is contained in both $W^{k,p}_\lambda$ and C^l_λ since stable maps are C^∞ ([47, Theorem 2.1]).

6.2.4. *Virtual dimension.* Let $\rho=[(\Sigma,\mathbf{B};\mathbf{p};\mathbf{q}^1,\ldots,\mathbf{q}^h;u)]\in M_\lambda$. Let $C_1,\ldots,C_\nu$ be irreducible components of Σ which are (possibly nodal) Riemann surfaces, and let $\Sigma_1,\ldots,\Sigma_{\nu'}$ be the remaining irreducible components of Σ, which are (possibly nodal) bordered Riemann surfaces. Let $\hat{C}_i$ denote the normalization of C_i for $i=1,\ldots,\nu$, let $\hat{\Sigma}_{i'}$ denote the normalization of $\Sigma_{i'}$ for $i'=1,\ldots,\nu'$, and $\tau:\hat{\Sigma}\to\Sigma$ be the normalization map.

Let $r_1,\ldots,r_{l_0}\in\Sigma^\circ$ be interior nodes of Σ, and $s_1,\ldots,s_{l_1}\in\partial\Sigma$ be boundary nodes of Σ. Let $\hat{p}_j\in\hat{\Sigma}$ be the preimage of p_j under μ for $j=1,\ldots,n$, $\hat{q}_{j'}$ be the preimage of $q_{j'}$ under τ for $j'=1,\ldots,m$, $\hat{r}_\alpha,\hat{r}_{l_0+\alpha}$ be the preimages of r_α under μ for $\alpha=1,\ldots,l_0$, and $\hat{s}_{\alpha'},\hat{s}_{l_1+\alpha'}$ be the preimages of $s_{\alpha'}$ under τ for $\alpha'=1,\ldots,l_1$.

Set $\hat{u}=u\circ\tau$. Then $\hat{u}:(\hat{\Sigma},\partial\hat{\Sigma})\to(X,L)$ is J-holomorphic. For $l>1$, let

$$C^l(\hat{\Sigma},\partial\hat{\Sigma},\hat{u}^*TX,(\hat{u}|_{\partial\hat{\Sigma}})^*TL)$$

denote the vector space of C^l sections of $\hat{u}^*TX$ with boundary values in $(\hat{u}|_{\partial\hat{\Sigma}})^*TL$. Then

$$\begin{aligned}&C^l(\hat{\Sigma},\partial\hat{\Sigma},\hat{u}^*TX,(\hat{u}|_{\partial\hat{\Sigma}})^*TL)\\&=\bigoplus_{i=1}^{\nu}C^l(\hat{C}_i,(\hat{u}|_{\partial\hat{C}_i})^*TX)\oplus\bigoplus_{i'=1}^{\nu'}C^l(\hat{\Sigma}_{i'},\partial\hat{\Sigma}_{i'},(\hat{u}|_{\hat{\Sigma}_{i'}})^*TX,(\hat{u}|_{\partial\hat{\Sigma}_{i'}})^*TL)\end{aligned}$$

Let $C^l(\Sigma,\partial\Sigma,u^*TX,(u|_{\partial\Sigma})^*TL)$ be the kernel of

$$C^l(\hat{\Sigma},\partial\hat{\Sigma},\hat{u}^*TX,(\hat{u}|_{\partial\hat{\Sigma}})^*TL)\longrightarrow\bigoplus_{\alpha=1}^{l_0}T_{r_\alpha}X\oplus\bigoplus_{\alpha'=1}^{l_1}T_{s_{\alpha'}}L$$

$$s\mapsto\left(\{s(\hat{r}_\alpha)-s(\hat{r}_{l_0+\alpha})\}_{\alpha=1}^{l_0},\{s(\hat{s}_{\alpha'})-s(\hat{s}_{l_1+\alpha'})\}_{\alpha'=1}^{l_1}\right),$$

and set

$$\begin{aligned}&C^{l-1}(\Sigma,\Lambda^{0,1}\Sigma\otimes u^*TX)\\&=\quad\bigoplus_{i=1}^{\nu}C^{l-1}(\hat{C}_i,\Lambda^{0,1}\hat{C}_i\otimes(\hat{u}|_{\hat{C}_i})^*TX)\oplus\bigoplus_{i'=1}^{\nu'}C^{l-1}(\hat{\Sigma}_{i'},\Lambda^{0,1}\hat{\Sigma}_{i'}\otimes(\hat{u}|_{\hat{\Sigma}_{i'}})^*TX)\end{aligned}$$

The linearization of $\bar{\partial}_{J,\Sigma}$ at the map u gives rise to operators

$$C^l(\hat{C}_i,(\hat{u}|_{\hat{C}_i})^*TX)\to C^{l-1}(\hat{C}_i,\Lambda^{0,1}\hat{C}_i\otimes(\hat{u}|_{\hat{C}_i})^*TX),\quad i=1,\ldots,\nu,$$

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$$C^l(\hat{\Sigma}_{i'},\partial\hat{\Sigma}_{i'},(\hat{u}|_{\hat{\Sigma}_{i'}})^*TX,(\hat{u}|_{\partial\hat{\Sigma}_{i'}})^*TL)\to C^{l-1}(\hat{\Sigma}_{i'},\Lambda^{0,1}\hat{\Sigma}_{i'}\otimes(\hat{u}|_{\hat{\Sigma}_{i'}})^*TX)\ \ i'=1,\ldots,\nu',$$

and give the operator

$$D_u: C^l(\Sigma,\partial\Sigma,u^*TX,(u|_{\partial\Sigma})^*TL)\to C^{l-1}(\Sigma,\Lambda^{0,1}\Sigma\otimes u^*TX)$$

Similarly, we have the operator

$$D_u: W^{k,p}(\Sigma,\partial\Sigma,u^*TX,(u|_{\partial\Sigma})^*TL)\to W^{k-1,p}(\Sigma,\Lambda^{0,1}\Sigma\otimes u^*TX).$$

The following proposition is proved by a straightforward computations.

Proposition 6.12. *Let ∇ be a connection on TX. Then*

$$D_u(\xi)=\frac{1}{2}\left(\nabla\xi\circ du+J\circ\nabla\xi\circ du\circ j+\nabla_\xi J\circ du\circ j+T(\xi,du)+JT(\xi,du\circ j)\right)$$

where $T(v,w)=\nabla_v w-\nabla_w v-[v,w]$ is the torsion of ∇.

Lemma 6.13. *Let Σ be a smooth bordered Riemann surface of type (g,h), and $u:(\Sigma,\partial\Sigma)\to(X,L)$ be a J-holomorphic map. Then*

$$D_u: C^\infty(\Sigma,\partial\Sigma,u^*TX,(u|_{\partial\Sigma})^*TL)\to C^\infty(\Sigma,\Lambda^{0,1}\Sigma\otimes u^*TX)$$

is a Fredholm operator of index $\mu+N(2-2g-h)$, where $\mu=\mu(u^*TX,(u|_{\partial\Sigma})^*TL)$, and $2N=\dim_{\mathbb{R}}X$.

Proof. Let ∇ be the Levi-Civita connection. ∇ is torsion-free, hence by Proposition 6.12,

$$D_u(\xi)=\frac{1}{2}\left(\nabla\xi\circ du+J\circ\nabla\xi\circ du\circ j+\nabla_\xi J\circ du\circ j\right).$$

We have $D_u=D''+R$, where

$$\begin{aligned} D''(\xi) &= \frac{1}{2}(\nabla\xi\circ du+J\circ\nabla\xi\circ du\circ j)\\ R(\xi) &= =\frac{1}{2}\nabla_\xi J\circ du\circ j \end{aligned}$$

D'' defines a holomorphic structure on u^*TX such that $D''=\bar{\partial}$. By [24, Theorem 3.4.2], D'' is a Fredholm operator of index $\mu+N(2-2g-h)$. R is a compact operator, so $D_u=D''+R$ is Fredholm of index $\mu+N(2-2g-h)$. □

Proposition 6.14. *Let $u:(\Sigma,\partial\Sigma)\to(X,L)$ be a prestable map. Then*

$$D_u: C^\infty(\Sigma,\partial\Sigma,u^*TX,(u|_{\partial\Sigma})^*TL)\to C^\infty(\Sigma,\Lambda^{0,1}\Sigma\otimes u^*TX)$$

is a Fredholm operator of index $\mu+N(1-\tilde{g})$, where $\mu=\mu(u^*TX,(u|_{\partial\Sigma})^*TL)$, and $\tilde{g}$ is the arithmetic genus of $\Sigma_{\mathbb{C}}$.

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Proof. We use the above notation. Set

$$\begin{aligned}
C^0 &= C^\infty(\Sigma, \partial\Sigma, u^*TX, (u|_{\partial\Sigma})^*TL)\\
C^1 &= C^\infty(\Sigma, \Lambda^{0,1}\Sigma\otimes u^*TX)\\
\tilde{C}_i^0 &= C^\infty(\hat{C}_i, (\hat{u}|_{\hat{C}_i})^*TX)\\
\tilde{C}_i^1 &= C^\infty(\hat{C}_i, \Lambda^{0,1}\hat{C}_i\otimes(\hat{u}|_{\hat{C}_i})^*TX)\\
C_{i'}^0 &= C^\infty(\hat{\Sigma}_{i'}, \partial\hat{\Sigma}_{i'}, (\hat{u}|_{\hat{\Sigma}_{i'}})^*TX, (\hat{u}|_{\partial\hat{\Sigma}_{i'}})^*TL)\\
C_{i'}^1 &= C^\infty(\hat{\Sigma}_{i'}, \Lambda^{0,1}\hat{\Sigma}_{i'}\otimes(\hat{u}|_{\hat{\Sigma}_{i'}})^*TX)
\end{aligned}$$

where $i=1,\dots,\nu$, $i'=1,\dots,\nu'$. The linearization of $\bar{\partial}_{J,\Sigma}$ gives rise to Fredholm operators $\tilde{D}_i : \tilde{C}_i^0 \to \tilde{C}_i^1$ for $i=1,\dots,\nu$ and $D_{i'} : C_{i'}^0 \to C_{i'}^1$ for $i'=1\dots,\nu'$. We have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & C^0 \arrow[r] \arrow[d, "D_u"'] & \bigoplus_{i=1}^{\nu}\tilde{C}_i^0 \oplus \bigoplus_{i'=1}^{\nu'} C_{i'}^0 \arrow[r] \arrow[d, "D"'] & \bigoplus_{\alpha=1}^{l_0} T_{r_\alpha}X \oplus \bigoplus_{\alpha'=1}^{l_1} T_{s_{\alpha'}}L \arrow[r] \arrow[d] & 0\\
0 \arrow[r] & C^1 \arrow[r] & \bigoplus_{i=1}^{\nu}\tilde{C}_i^1 \oplus \bigoplus_{i'=1}^{\nu'} C_{i'}^1 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

where $D = \bigoplus_{i=1}^{\nu}\tilde{D}_i \oplus \bigoplus_{i'=1}^{\nu'} D_{i'}$, and the rows are exact. So D_u is Fredholm.

Given a Fredholm operator D, let $\mathrm{Ind}(D)$ denote the virtual real vector space

$$\mathrm{Ker}(D) - \mathrm{Coker}(D),$$

whose dimension

$$\dim \mathrm{Ind}(D) = \dim \mathrm{Ker}(D) - \dim \mathrm{Coker}(D)$$

is the Fredholm index of D. With the above notation, we have

$$\dim \mathrm{Ind}(D_u) = \sum_{i=1}^{\nu} \dim \mathrm{Ind}(\tilde{D}_i) + \sum_{i'=1}^{\nu'} \dim \mathrm{Ind}(D_{i'}) - 2Nl_0 - Nl_1.$$

Suppose that $\hat{C}_i$ is of genus $\hat{g}_i$, and $\hat{\Sigma}_{i'}$ is of type $(g_{i'}, h_{i'})$. We have

$$\begin{aligned}
\dim \mathrm{Ind}(\tilde{D}_i) &= 2\deg((\hat{u}|_{\hat{C}_i})^*TX) + 2N(1-\hat{g}_i),\\
\dim \mathrm{Ind}(D_{i'}) &= \mu((\hat{u}|_{\hat{\Sigma}_{i'}})^*TX, (\hat{u}|_{\partial\hat{\Sigma}_{i'}})^*TL) + N(2-2g_{i'}-h_{i'}),
\end{aligned}$$

where the second equality follows from Proposition 6.13. $\Sigma_{\mathbb{C}}$ has $2l_0+l_1$ nodes and $2\nu+\nu'$ irreducible components

$$C_1,\dots,C_\nu,\ \bar{C}_1,\dots,\bar{C}_\nu,\ (\Sigma_1)_{\mathbb{C}},\dots,(\Sigma_{\nu'})_{\mathbb{C}},$$

where the genus of $(\hat{\Sigma}_{\nu'})_{\mathbb{C}}$ is $\tilde{g}_{i'} = 2g_{i'}+h_{i'}-1$, so the arithmetic genus of $\Sigma_{\mathbb{C}}$ is

$$\begin{aligned}
\tilde{g} &= 2\sum_{i=1}^{\nu}\hat{g}_i + \sum_{i'=1}^{\nu'}\tilde{g}_{i'} + 2l_0 + l_1 - 2\nu - \nu' + 1\\
&= 2\sum_{i=1}^{\nu}(\hat{g}_i - 1) + \sum_{i'=1}^{\nu'}(2g_{i'}+h_{i'}-2) + 2l_0 + l_1 + 1
\end{aligned}$$

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by [18, (3.1)]. Finally,

$$\mu=\mu(u^*TX,(u|_{\partial\Sigma})^*TL)=2\sum_{i=1}^{\nu}\deg((\hat{u}|_{\hat{C}_i})^*TX+\sum_{i'=1}^{\nu'}\mu((\hat{u}|_{\hat{\Sigma}_{i'}})^*TX,(\hat{u}|_{\partial\hat{\Sigma}_{i'}})^*TL).$$

We conclude that

$$\dim\operatorname{Ind}(D_u)=\mu(u^*TX,(u|_{\partial\Sigma})^*TL)+N(1-\tilde{g}).$$

□

Remark 6.15. *Corollary 6.14 remains true for*

$$D_u:W^{k,p}(\Sigma,\partial\Sigma,u^*TX,(u|_{\partial\Sigma})^*TL)\to W^{k-1,p}(\Sigma,\Lambda^{0,1}\Sigma\otimes u^*TX)$$
$$D_u:C^{l}(\Sigma,\partial\Sigma,u^*TX,(u|_{\partial\Sigma})^*TL)\to C^{l-1}(\Sigma,\Lambda^{0,1}\Sigma\otimes u^*TX)$$

Set $\mathcal{E}^\infty_u=C^\infty(\Sigma,\Lambda^{0,1}\Sigma\otimes u^*TX)$ for $u\in C^\infty_\lambda$. The $\mathcal{E}^\infty_u$ fit together to form a Banach orbibundle $\mathcal{E}^\infty_\lambda\to C^\infty_\lambda$. There is a section $s_J:C^\infty_\lambda\to\mathcal{E}^\infty_\lambda$, defined by $u\mapsto\bar{\partial}_{J,\Sigma}u$, and M_λ is the zero locus of s_J. If λ has no nontrivial automorphism, then C^∞_λ is a Banach manifold, and $\mathcal{E}^\infty_\lambda\to C^\infty_\lambda$ is a Banach bundle. In this case, if M_λ is nonempty and D_u is surjective for all $u\in M_\lambda$, then M_λ is a smooth manifold of dimension $\mu+N(2-2g-h)$ by the implicit function theorem. We call $\mu+N(2-2g-h)$ the *virtual* dimension of M_λ. In general, M_λ is singular, and the actual dimension of M_λ can be larger than the virtual dimension.

The dimension of $\widetilde{M}_{(g,h),(n,\vec{m})}$ is $6g+3h-6+2n+m^1+\cdots+m^h$, so the expected (or virtual) dimension of $\overline{M}_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)$ is

$$\mu+(N-3)(2-2g-h)+2n+m^1+\cdots+m^h,$$

which is the virtual dimension of the Kuranishi structure on $\overline{M}_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)$.

Similarly, we have

$$\mathcal{E}^l_\lambda\to C^l_\lambda,\qquad s_J:C^l_\lambda\to\mathcal{E}^l_\lambda,$$
$$\mathcal{E}^{k,p}_\lambda\to W^{k,p}_\lambda,\qquad s_J:W^{k,p}_\lambda\to\mathcal{E}^{k,p}_\lambda,$$

and M_λ is the zero locus of s_J in the above spaces.

6.3. **Deformation of the domain.** Let

$$\rho=[(\Sigma,\mathbf{B};\mathbf{p};\mathbf{q}^1,\ldots,\mathbf{q}^h;u)]$$

be a point in $\overline{M}_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)$. We have seen in Section 3.3 that the infinitesimal deformation of the domain

$$\lambda=[(\Sigma,\mathbf{B};\mathbf{p};\mathbf{q}^1,\ldots,\mathbf{q}^h)]$$

is given by

$$H_{\rho,\mathrm{domain}}=\bigoplus_{i=1}^{\nu}W_i\oplus\bigoplus_{i'=1}^{\nu'}\hat{W}_{i'}\oplus\bigoplus_{\alpha=1}^{l_0}V_\alpha\oplus\bigoplus_{\alpha'=1}^{l_1}\hat{V}^+_{\alpha'},$$

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where

$$\begin{aligned}
W_i &= H^1(\hat{C}_i, T_{\hat{C}_i}(-D_{\mathbf{y}^i})),\\
\hat{W}_{i'} &= H^1(\hat{\Sigma}_{i'}, \partial\hat{\Sigma}_{i'}, T_{\hat{\Sigma}_{i'}}(-p_1^{i'} - \cdots - p_{n_{i'}}^{i'}), T_{\partial\hat{\Sigma}_{i'}}(-q_1^{i'} - \ldots - q_{m_{i'}}^{i'})),\\
V_\alpha &= T_{\hat{r}_\alpha}\hat{\Sigma} \otimes T_{\hat{r}_{l_0+\alpha}}\hat{\Sigma} \cong \mathbb{C},\\
\hat{V}_{\alpha'} &= T_{\hat{s}_{\alpha'}}\partial\hat{\Sigma} \otimes T_{\hat{s}_{l_1+\alpha'}}\partial\hat{\Sigma} \cong \mathbb{R},\\
\hat{V}_{\alpha'}^+ &\cong [0,\infty) \subset \hat{V}_{\alpha'}
\end{aligned}$$

are defined as in Section 3.3. $\hat{V}_{\alpha'}^+$ is only a semigroup, while the others are vector spaces. Set

$$\begin{aligned}
H_{\rho,\text{deform}} &= \bigoplus_{i=1}^{\nu} W_i \oplus \bigoplus_{i'=1}^{\nu'} \hat{W}_{i'}\\
H_{\rho,\text{interior}} &= \bigoplus_{\alpha=1}^{l_0} V_\alpha\\
H_{\rho,\text{boundary}} &= \hat{V}_1^+ \times \cdots \times \hat{V}_{l_1}^+\\
H_{\rho,\text{smooth}} &= H_{\rho,\text{interior}} \times H_{\rho,\text{boundary}}
\end{aligned}$$

Then $H_{\rho,\text{deform}}$ corresponds to tangent directions of the stratum to which λ belongs, while $H_{\rho,\text{smooth}}$ corresponds to normal directions to this stratum. $H_{\rho,\text{interior}}$ corresponds to smoothing of interior nodes, and $H_{\rho,\text{boundary}}$ corresponds to smoothing of boundary nodes.

Let $(\operatorname{Aut}\lambda)_0$ denote the identity component of $\operatorname{Aut}\lambda$. $(\operatorname{Aut}\lambda)_0$ is a normal subgroup of $\operatorname{Aut}\lambda$, and the quotient $\operatorname{Aut}'\lambda = \operatorname{Aut}\lambda/(\operatorname{Aut}\lambda)_0$ is a finite group. $\operatorname{Aut}\lambda$ acts on $H_{\rho,\text{deform}}$, and $(\operatorname{Aut}\lambda)_0$ acts trivially, so $\operatorname{Aut}'\lambda$ acts on $H_{\rho,\text{deform}}$.

We choose an admissible metric h on Σ in the sense of Section 6.2.2. Let ϵ_1 be a small positive number, and define $N_{\epsilon_1}(\Sigma)$, $K_{\epsilon_1}(\Sigma)$ as in the paragraph right before Definition 5.5. Then $N_{\epsilon_1}(\Sigma)$, $K_{\epsilon_1}(\Sigma)$ are invariant under $\operatorname{Aut}\rho$. We may choose a subspace $\tilde{H}$ of the space of smooth Beltrami differentials such that the elements in $\tilde{H}$ vanish on $N_{\epsilon_1}(\Sigma)$, and the natural map $\tilde{H} \to H_{\rho,\text{deform}}$ is an isomorphism. We may further assume that $\tilde{H}$ is invariant under the action of $\operatorname{Aut}'\rho$, so that $\operatorname{Aut}'\rho$ acts on $\tilde{H}$ and the isomorphism $\tilde{H} \to H_{\rho,\text{deform}}$ is $\operatorname{Aut}'\rho$-equivariant. From now on, we will identify $\tilde{H}$ with $H_{\rho,\text{deform}}$.

6.3.1. *Deformation within the stratum.* Let $j(\xi)$ be the complex structure on Σ determined by $\xi \in \tilde{H}$, and let $\Sigma_{(\xi,0,0)}$ be the prestable bordered Riemann surface corresponding to $(\Sigma, j(\xi))$. In particular, $j(0)$ is the original complex structure j on Σ, and $\Sigma_{(0,0,0)} = \Sigma$. Set

$$\lambda_{(\xi,0,0)} = (\Sigma_{(\xi,0,0)}, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h).$$

Let $\kappa_{(\xi,0,0)} : \Sigma_{(\xi,0,0)} \to \Sigma$ be the identity map. Then

(1) $\kappa_{(\xi,0,0)} : \lambda_{(\xi,0,0)} \to \lambda$ is a strong deformation in the sense of Definition 4.8. $\kappa_{(\xi,0,0)} : \Sigma_{(\xi,0,0)} \to \Sigma$ is a homeomorphism.
(2) $j = \left(\kappa_{(\xi,0,0)}^{-1}\right)^* j(\xi)$ on $N_{\epsilon_1}(\Sigma)$.

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(3) $\| j - \left(\kappa^{-1}_{(\xi,0,0)}\right)^* j(\xi) \|_{C^\infty(K_{\epsilon_1}(\Sigma))} < C|\xi|$, where $|\xi|$ is the Weil-Petersson norm of the Beltrami differential ξ.

Note that any two norms on $H_{\rho,\mathrm{deform}}$ are equivalent since $H_{\rho,\mathrm{deform}}$ is finite dimensional. Let $B_{\delta_2} \subset H_{\rho,\mathrm{deform}}$ be the ball of radius $\delta_2 > 0$ centered at the origin. From the above discussion, we see that there is a family of prestable bordered Riemann surfaces of type (g,h) with $(n,\vec{m})$ marked points $\{\lambda_{(\xi,0,0)} \mid \xi \in B_{\delta_2}\}$. More precisely, we have

$$(\pi : \mathcal{C} \to B_{\delta_2}; \mathbf{s}; \mathbf{t}^1, \ldots, \mathbf{t}^h),$$

where $\mathbf{s} = (s_1, \ldots, s_n)$, $\mathbf{t}^i = (t^i_1, \ldots, t^i_{m^i})$, and a contraction $\kappa : \mathcal{C} \to \Sigma$. Diffeomorphically, $\mathcal{C} = B_{\delta_2} \times \Sigma$, π is the projection to the first factor, κ is the projection of the second factor, $s_j, t^i_k : B_{\delta_2} \to \mathcal{C}$ are constant sections corresponding to marked points p_j, q^i_k, respectively. Holomorphically, $\pi^{-1}(\xi) = \Sigma_{(\xi,0,0)}$.

Let ω_0 be the volume form on Σ determined by h. Then ω_0 is a Kähler form. Let $h_{(\xi,0,0)}$ be the Hermitian metric on $\Sigma_{(\xi,0,0)}$ determined by $\kappa^*_{(\xi,0,0)}\omega_0$ and $j(\xi)$. Then $\kappa_{(\xi,0,0)}$ is an isometry near the boundary and nodes, so $h_{(\xi,0,0)}$ is an admissible metric on $\Sigma_{(\xi,0,0)}$.

There is a map $i : B_{\delta_2} \to \widetilde{M}_{(g,h),(n,\vec{m})}$, given by $\xi \mapsto \lambda_{(\xi,0,0)}$. Given $\epsilon_2 > 0$, there exists $\delta_2 > 0$ such that $i(B_{\delta_2}) \subset U(\lambda, \epsilon_1, \epsilon_2)$, where $U(\lambda, \epsilon_1, \epsilon_2)$ is the neighborhood of λ in $\widetilde{M}_{(g,h),(n,\vec{m})}$ in the C^∞ topology defined in Definition 5.6. $i(B_{\delta_2})$ is a neighborhood of λ in the stratum of $\widetilde{M}_{(g,h),(n,\vec{m})}$ to which λ belongs.

6.3.2. *Smoothing of interior nodes.* Let s be an interior node of Σ, and let $i : B_{\epsilon_1}(s) \to \mathbb{C}^2$ be a holomorphic isometry such that $i(B_{\epsilon_1}(s)) = \{(x,y) \in \mathbb{C}^2 \mid xy = 0, |x| < \epsilon_1, |y| < \epsilon_1\}$. Let $s_1, s_2 \in \hat{\Sigma}$ be the preimages of s under $\tau : \hat{\Sigma} \to \Sigma$. Up to permutation of s_1, s_2, there exist unique $e_1 \in T_{s_1}\hat{\Sigma}$, $e_2 \in T_{s_2}\hat{\Sigma}$ such that $(i \circ \tau)_*(e_1) = (1,0) \in \mathbb{C}^2$, and $(i \circ \tau)_*(e_2) = (0,1) \in \mathbb{C}^2$.

Given $v \in T_{s_1}\hat{\Sigma} \otimes T_{s_2}\hat{\Sigma}$, we have $v = te_1 \otimes e_2$ for some $t \in \mathbb{C}$. Suppose that $t = r^2 e^{i\phi}$, where $0 < r < \frac{\epsilon_1}{3}$. Let Σ_t be the bordered Riemann surface obtained from Σ by replacing

$$B_{\epsilon_1,0} = \{(x,y) \in \mathbb{C}^2 \mid xy = 0, |x| < \epsilon_1, |y| < \epsilon_1\}$$

with

$$B_{\epsilon_1,t} = \{(x,y) \in \mathbb{C}^2 \mid xy = t, |x| < \epsilon_1, |y| < \epsilon_1\}$$

More precisely, $B_{\epsilon_1,t}$ is obtained by identifying

$$x = re^{i\theta} \in \{x \in \mathbb{C} \mid |x| = r\} \subset \{x \in \mathbb{C} \mid r \le |x| < \epsilon_1\}$$

with

$$\frac{t}{x} = re^{i(\phi-\theta)} \in \{y \in \mathbb{C} \mid |x| = r\} \subset \{y \in \mathbb{C} \mid r \le |y| < \epsilon_1\}.$$

There is a strong deformation

$$\kappa_t : B_{\epsilon_1,t} = \left\{ \left(x, \frac{t}{x}\right) \,\middle|\, \frac{r^2}{\epsilon_1} < |x| < \epsilon_1 \right\} \to B_{\epsilon_1,0}$$

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given by

$$\kappa_t\left(x, \frac{t}{x}\right) = \begin{array}{ll} \Big(\chi(\left|\frac{x}{r}\right|^2)x, 0\Big) & \text{if } r \leq |x| < \epsilon_1 \\ \Big(0, \chi(\left|\frac{t}{xr}\right|^2)\frac{t}{x}\Big) & \text{if } \dfrac{r^2}{\epsilon_1} < |x| \leq r \end{array}$$

where $\chi : \mathbb{R} \to [0,1]$ is a smooth function such that

(1) $0 \leq \chi'(s) \leq 1$.
(2) $\chi(s) = 0$ for $s \leq 1$.
(3) $\chi(s) > 0$ for $s > 1$.
(4) $\chi(s) = 1$ for $s \geq 4$.

Lemma 6.16. *Let f, g be smooth functions on $\mathbb{C}$ such that $f(0) = g(0)$. Let F be the continuous function on $\{z \in \mathbb{C} \mid \frac{r^2}{\epsilon_1} < |z| < \epsilon_1\}$ defined by*

$$F(z) = \begin{cases} f(\chi(\left|\frac{z}{r}\right|^2)z) & \text{if } r \leq |z| < \epsilon_1 \\ g\left(\chi(\left|\frac{t}{zr}\right|^2)\frac{t}{z}\right) & \text{if } \frac{r^2}{\epsilon_1} < |z| \leq r \end{cases}$$

Then F is smooth.

Proof. The lemma follows from

(1) Both $h_1(z) = f(\chi(\left|\frac{z}{r}\right|^2)z)$, $h_2(z) = g\left(\chi(\left|\frac{t}{zr}\right|^2)\frac{t}{z}\right)$ are smooth for $\frac{r^2}{\epsilon_1} < |z| < \epsilon_1$.
(2) The derivatives of h_1 of any order vanish when $|z| = r$, and the same is true for h_2.

□

Let $A(r,R)$ denote the annulus $\{(u,v) \in \mathbb{R}^2 \mid r^2 \leq u^2+v^2 \leq R^2\}$, and let (r,R) denote its interior. Let D_R denote the closed disc $\{(u,v) \in \mathbb{R}^2 \mid u^2+v^2 \leq R^2\}$.

Lemma 6.17. *Let f, g, F be defined as in Lemma 6.16. Define $h : (\frac{r^2}{\epsilon_1}, \epsilon_1) \to \mathbb{C}$ by $h(u,v) = F(u+iv)$. Then*

$$\begin{aligned} \max_{A(\frac{r}{2},2r)} |h| &= \max\{\max_{D_{2r}} |f|, \max_{D_{2r}} |g|, \} \\ \max_{A(\frac{r}{2},2r)} |\nabla h| &\leq 9\sqrt{2}\max\{\max_{D_{2r}} |f'|, 4\max_{D_{2r}} |g'|\}, \end{aligned}$$

where $|\nabla h|^2 = |h_u|^2 + |h_v|^2$.

Proof. We have

$$h(u,v) = \begin{cases} f(\chi(\frac{u^2+v^2}{r^2})(u+iv)) & \text{if } r^2 \leq u^2+v^2 < \epsilon_1^2, \\ g\left(\chi(\frac{r^2}{u^2+v^2})\frac{t}{u+iv}\right) & \text{if } \frac{r^4}{\epsilon_1^2} < u^2+v^2 \leq r^2, \end{cases}$$

thus

$$\sup_{A(\frac{r}{2},2r)} |h| = \max\{\max_{D_{2r}} |f|, \max_{D_{2r}} |g|\}.$$

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We also have

$$h_u(u,v)=\begin{cases} f'(\chi(\frac{u^2+v^2}{r^2})(u+iv))(\chi'(\frac{u^2+v^2}{r^2})\frac{2u(u+iv)}{r^2}+\chi(\frac{u^2+v^2}{r^2})) & \text{if } r^2<u^2+v^2<\epsilon_1^2,\\ g'(\chi(\frac{r^2}{u^2+v^2})\frac{t}{u+iv})(\chi'(\frac{r^2}{u^2+v^2})\frac{2r^2u}{(u^2+v^2)^2}\frac{t}{u+iv}-\chi(\frac{r^2}{u^2+v^2})\frac{t}{(u+iv)^2}) & \text{if } \frac{r^4}{\epsilon_1^2}<u^2+v^2<r^2,\end{cases}$$

and $h_u(u,v)=0$ for $u^2+v^2=r^2$, hence

$$\sup_{A(\frac{r}{2},2r)}|h_u|\leq \max\{9\max_{D_{2r}}|f'|,36\max_{D_{2r}}|g'|.\}$$

Similarly,

$$\sup_{A(\frac{r}{2},2r)}|h_v|\leq \max\{9\max_{D_{2r}}|f'|,36\max_{D_{2r}}|g'|.\}$$

□

In particular, let $(f(x),g(y))=(x,0),(0,y)$. We see that κ_t is smooth as a map to $\mathbb{C}^2$. κ_t is a diffeomorphism when $|x|\neq r$, and $\kappa_t^{-1}(0,0)=\{(x,\frac{t}{x})\mid |x|=r\}$. Choose a Hermitian metric h_t on $B_{\epsilon_1,t}$ such that it is induced by inclusion in $\mathbb{C}^2$ on $B_{2r,t}$ and κ_t is an isometry outside $B_{3r,t}$.

We now have a family of prestable bordered Riemann surfaces of type (g,h) with $(n,\vec{m})$ marked points

$$\lambda_t=(\Sigma_t,\mathbf{B};\mathbf{p};\mathbf{q}^1,\ldots,\mathbf{q}^h)$$

together with a family of admissible metrics h_t on Σ_t such that

(1) There are strong deformations $\kappa_t:\lambda_t\to\lambda$ such that on $K_{3r}(\Sigma)$, where $r=\sqrt{|t|}$, κ_t^{-1} is defined and is an isometry.

(2) $j=\left(\kappa_t^{-1}\right)^*j_t$ on $K_{3r}(\Sigma)$, where j_t is the complex structure on Σ_t.

Let $D_{\epsilon_1^2/9}=\{te_1\otimes e_2\mid |t|<\epsilon_1^2/9\}\subset T_{s_1}\hat{\Sigma}\otimes T_{s_2}\hat{\Sigma}$. The map $D_{\epsilon_1^2/9}\to\widetilde{M}_{(g,h),(n,\vec{m})}$ given by $te_1\otimes e_2\mapsto\lambda_t$ defines a parametrized curve in $\widetilde{M}_{(g,h),(n,\vec{m})}$ whose tangent line at $\lambda_0=\lambda$ is $T_{s_1}\hat{\Sigma}\otimes T_{s_2}\hat{\Sigma}\subset H_{\rho,\mathrm{interior}}$.

Let $\eta=(v_1,\ldots,v_{l_0})\in H_{\rho,\mathrm{interior}}$, where $v_\alpha\in V_\alpha=T_{r_\alpha}\hat{\Sigma}\otimes T_{r_{l_0+\alpha}}\hat{\Sigma}$. Applying the above construction to each interior node on $\Sigma_{(\xi,0,0)}$, we obtain

$$\lambda_{(\xi,\eta,0)}=(\Sigma_{(\xi,\eta,0)},\mathbf{B};\mathbf{p};\mathbf{q}^1,\ldots,\mathbf{q}^h).$$

Given $0<d_1,\ldots,d_{l_0}<\epsilon_1^2/9$, let $D(d_1,\ldots,d_{l_0})$ denote the polydisc $D_{d_1}\times\ldots\times D_{d_{l_0}}$ in $H_{\rho,\mathrm{interior}}$, where D_{d_α} is the disc of radius d_α centered at the origin in V_α. We have a family

$$(\pi:\mathcal{C}\to B_{\delta_2}\times D(d_1,\ldots,d_{l_0});\mathbf{s};\mathbf{t}^1,\ldots,\mathbf{t}^h)$$

of prestable bordered Riemann surfaces of type (g,h) with $(n,\vec{m})$ marked points together with a family of admissible metrics $h_{(\xi,\eta,0)}$ on $\Sigma_{(\xi,\eta,0)}$. There is a contraction $\kappa:\mathcal{C}\to\Sigma$ whose restriction to $\pi^{-1}(\xi,\eta)=\lambda_{(\xi,\eta,0)}$ is a strong deformation $\kappa_{(\xi,\eta,0)}:\lambda_{(\xi,\eta,0)}\to\lambda$ such that

(1) $\kappa^{-1}_{(\xi,\eta,0)}$ is defined on $K_{3\sqrt{|\eta|}}(\Sigma)$, and $\kappa^{-1}_{(\xi,\eta,0)}\circ\kappa_{(\xi,0,0)}$ is an isometry on $K_{3\sqrt{|\eta|}}(\Sigma_{(\xi,0,0)})$.

(2) $\|\, j-\left(\kappa^{-1}_{(\xi,\eta,0)}\right)^*j(\xi,\eta)\,\|_{C^\infty(K_{\epsilon_1}(\Sigma))}=\|\, j-\left(\kappa^{-1}_{(\xi,0,0)}\right)^*j(\xi)\,\|_{C^\infty(K_{\epsilon_1}(\Sigma))}<C|\xi|$, where $j(\xi,\eta)$ is the complex structure on $\Sigma_{(\xi,\eta,0)}$.

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6.3.3. *Smoothing of boundary nodes.* Let s be a boundary node of Σ, and let $i : B_{\epsilon_1}(s) \to \mathbb{C}^2$ be a holomorphic isometry such that

$$i(B_{\epsilon_1}(s)) = \{(x,y) \in \mathbb{C}^2 \mid xy = 0, |x| < \epsilon_1, |y| < \epsilon_1, \operatorname{Im} x \geq 0, \operatorname{Im} y \leq 0\},$$

Let $s_1, s_2 \in \hat{\Sigma}$ be the preimages of s under $\tau : \hat{\Sigma} \to \Sigma$. Up to permutation of s_1, s_2, there exist unique $e_1 \in T_{s_1}\hat{\Sigma}$, $e_2 \in T_{s_2}\hat{\Sigma}$ such that $(i \circ \tau)_*(e_1) = (1,0) \in \mathbb{C}^2$, and $(i \circ \tau)_*(e_2) = (0,1) \in \mathbb{C}^2$.

Given $v \in T_{s_1}\partial\hat{\Sigma} \otimes T_{s_2}\partial\hat{\Sigma}$, we have $v = te_1 \otimes e_2$ for some $t \in \mathbb{R}$. We construct λ_t as in Section 6.3.2. We have seen in Section 3.3 that there is topological transition when t changes sign. We may assume that $\lambda_t \in \widetilde{M}_{(g,h),(n,\vec{m})}$ for $t \geq 0$.

Let $\eta' = (v'_1, \ldots, v'_{l_0}) \in H_{\rho,\text{boundary}}$, where $v'_{\alpha'} \in \hat{V}^+_{\alpha'}$. Applying the above construction to each boundary node on $\Sigma_{(\xi,\eta,0)}$, we obtain $\Sigma_{(\xi,\eta,\eta')}$ and $\lambda_{(\xi,\eta,\eta')}$. Given $0 < d'_1, \ldots, d'_{l_1} < \epsilon_1^2/9$, let $D'(d'_1, \ldots, d'_{l_0}) = [0, d'_1) \times \ldots \times [0, d'_{l_1}) \subset \hat{V}_1 \times \ldots \times \hat{V}_{l_1}$. We have a universal family

$$(\pi : \mathcal{C} \to B_{\delta_2} \times D(d_1, \ldots, d_{l_0}) \times D'(d'_1, \ldots, d'_{l_1}); \mathbf{s}; \mathbf{t}^1, \ldots, \mathbf{t}^h)$$

of prestable bordered Riemann surfaces of type (g,h) with $(n,\vec{m})$ marked points together with a family of admissible metrics $h_{(\xi,\eta,\eta')}$ on $\Sigma_{(\xi,\eta,\eta')}$. There is a contraction $\kappa : \mathcal{C} \to \Sigma$ whose restriction to $\pi^{-1}(\xi,\eta,\eta')$ is a strong deformation $\kappa_{(\xi,\eta,\eta')} : \lambda_{(\xi,\eta,\eta')} \to \lambda$ such that

(1) $\kappa^{-1}_{(\xi,\eta,\eta')}$ is defined on $K_{3\sqrt{|\eta|+|\eta'|}}(\Sigma)$, and $\kappa^{-1}_{(\xi,\eta,\eta')} \circ \kappa_{(\xi,0,0)}$ is an isometry on $K_{3\sqrt{|\eta|+|\eta'|}}(\Sigma_{(\xi,0,0)})$.
(2) $\parallel j - \left(\kappa^{-1}_{(\xi,\eta,\eta')}\right)^* j' \parallel_{C^\infty(K_{\epsilon_1}(\Sigma))} < C|\xi|$.

If we embed $\Sigma_{\mathbb{C}}$ in a complex projective space $\mathbf{P}^N$, then $\kappa_{(\xi,\eta,\eta')} \circ \tau$ is smooth as a map to $\mathbf{P}^N$, where $\tau : \hat{\Sigma}_{(\xi,\eta,\eta')} \to \Sigma_{(\xi,\eta,\eta')}$ is the normalization map.

Given $\epsilon_1, \epsilon_2 > 0$, choose δ_2 as before. Suppose that both $\max\{\sqrt{|d_\alpha|} \mid \alpha = 1, \ldots, l_0\}$ and $\max\{\sqrt{|d'_{\alpha'}|} \mid \alpha' = 1, \ldots, l_1\}$ are less than $\frac{\epsilon_1}{3}$. Then the image of the map

$$i : B_{\delta_2} \times D(d_1, \ldots, d_{l_0}) \times D'(d'_1, \ldots, d'_{l_1}) \to \widetilde{M}_{(g,h),(n,\vec{m})}$$

given by $(\xi,\eta,\eta') \mapsto \lambda_{(\xi,\eta,\eta')}$ lies in the neighborhood $U(\lambda, \epsilon_1, \epsilon_2)$ of λ in the C^∞ topology.

6.3.4. *Action of the automorphism group.* We write D_d for $D(d_1, \ldots, d_{l_0})$ and $D'_{d'}$ for $D'(d'_1, \ldots, d'_{l_1})$. In this section, we study the action of $\operatorname{Aut}\lambda$ on $B_{\delta_2} \times D_d \times D'_{d'}$ and the universal family over it.

We first consider deformation within the stratum. Let $\pi_{\delta_2} : \mathcal{C}_{\delta_2} \to B_{\delta_2}$ be the universal family, so that $\pi^{-1}_{\delta_2}(\xi) = \lambda_{(\xi,0,0)}$. $\operatorname{Aut}\lambda$ acts on B_{δ_2} by $j(\phi \cdot \xi) = (\phi^{-1})^* j(\xi)$. Therefore, it acts on the the universal family. Given $\phi \in \operatorname{Aut}\lambda$, we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{C}_{\delta_2} \arrow[r, "\phi"] \arrow[d, "\pi_{\delta_2}"'] & \mathcal{C}_{\delta_2} \arrow[d, "\pi_{\delta_2}"] \\ B_{\delta_2} \arrow[r, "\phi"] & B_{\delta_2} \end{tikzcd}$$

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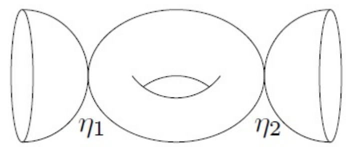

FIGURE 19. $\Sigma = \Sigma_{(0,0)}$

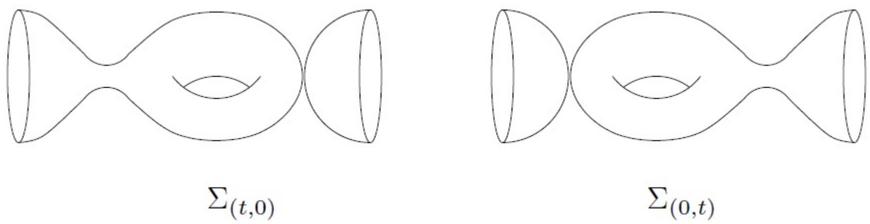

FIGURE 20. $\Sigma_{(t,0)}$ and $\Sigma_{(0,t)}$

$\phi : \lambda_{(\xi,0,0)} \to \lambda_{(\phi\cdot\xi,0,0)}$ is an isomorphism. In particular, if $\phi \in (\mathrm{Aut}\,\lambda)_0$, then $\phi : B_{\delta_2} \to B_{\delta_2}$ is the identity map, and we have the following commutative diagram:

$$\begin{tikzcd}
\lambda_{(\xi,0,0)} \arrow[r, "\phi"] \arrow[d, "\kappa_{(\xi,0,0)}"'] & \lambda_{(\xi,0,0)} \arrow[d, "\kappa_{(\xi,0,0)}"] \\
\lambda \arrow[r, "\phi"] & \lambda
\end{tikzcd}$$

We now consider smoothing of nodes. Let $\pi_{\delta_2,d,d'} : \mathcal{C}_{\delta_2,d,d'} \to B_{\delta_2} \times D_d \times D'_{d'}$ be the universal family, so that $\pi^{-1}_{\delta_2,d,d'}(\xi,\eta,\eta') = \lambda_{(\xi,\eta,\eta')}$. $\mathrm{Aut}\,\lambda$ acts on $D_d \times D'_{d'}$ by $\phi\cdot(te_1 \otimes e_2) = t\phi_* e_1 \otimes \phi_* e_2$. Given $\phi \in \mathrm{Aut}\,\lambda$, we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{C}_{\delta_2,d,d'} \arrow[r, "\phi"] \arrow[d, "\pi_{\delta_2,d,d'}"'] & \mathcal{C}_{\delta_2,d,d'} \arrow[d, "\pi_{\delta_2,d,d'}"] \\
B_{\delta_2} \times D_d \times D'_{d'} \arrow[r, "\phi"] & B_{\delta_2} \times D_d \times D'_{d'}
\end{tikzcd}$$

$\phi : \lambda_{(\xi,\eta,\eta')} \to \lambda_{(\phi\cdot\xi,\phi\cdot\eta,\phi\cdot\eta')}$ is an isomorphism. For example, the prestable bordered Riemann surface Σ in Figure 19 has two interior nodes. The smoothing of the two interior nodes is parametrized by $\eta = (\eta_1, \eta_2)$. Let $\Sigma_{(\eta_1,\eta_2)}$ be the corresponding bordered Riemann surfaces obtained by smoothing the two interior nodes on Σ.

Σ has an automorphism ϕ of order two which rotates Figure 19 by 180°. It acts on η by $\phi\cdot(\eta_1,\eta_2) = (\eta_2,\eta_1)$ and gives an isomorphism $\Sigma_{(\eta_1,\eta_2)} \to \Sigma_{(\eta_2,\eta_1)}$ by rotating 180°. The case $\eta_2 = 0$ is shown in Figure 20.

6.4. **Local Charts.** Let

$$\rho = [(\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h; u)]$$

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be a point in $\overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$, as at the beginning of Section 6.3. Consider

$$D_u : W^{1,p}(\Sigma, \partial\Sigma, u^*TX, (u|_{\partial\Sigma})^*TL) \to L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes u^*TX)$$

for a large integer p. $L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes u^*TX)$ is a complex vector space, but $W^{1,p}(\Sigma, \partial\Sigma, u^*TX, (u|_{\partial\Sigma})^*TL)$ is only a real vector space. $\mathrm{Aut}\,\rho$ acts on both $W^{1,p}(\Sigma, \partial\Sigma, u^*TX, (u|_{\partial\Sigma})^*TL)$ and $L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes u^*TX)$, and D_u is $\mathrm{Aut}\,\rho$-equivariant.

Lemma 6.18. *Let $\mathrm{Im}\, D_u$ denote the image of D_u. We can choose a subspace E_ρ of $L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes u^*TX)$ such that*

1. $\mathrm{Im}\, D_u + E_\rho = L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes u^*TX)$.
2. *E_ρ is finite dimensional.*
3. *Elements in E_ρ are smooth sections supported in $K_{\epsilon_1}(\Sigma)$.*
4. *E_ρ is a complex subspace of $L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes u^*TX)$.*
5. *E_ρ is $\mathrm{Aut}\,\rho$-invariant.*

Proof. We claim that given $\alpha \in L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes u^*TX)$, there exists $g \in W^{1,p}(\Sigma, \partial\Sigma, u^*TX, (u|_{\partial\Sigma})^*TL)$ such that $\alpha' = \alpha - D_u g$ has support in $K_{\epsilon_1}(\Sigma)$. Actually, there exists g' defined on $N_{2\epsilon_1}(\Sigma)$ such that $D_u g' = \alpha$ on $N_{2\epsilon_1}(\Sigma)$. Let χ be a smooth function on Σ which is 1 on $N_{\epsilon_1}(\Sigma)$ and 0 on $K_{2\epsilon_1}(\Sigma)$. Let $g = \chi g'$ on $N_{2\epsilon_1}(\Sigma)$, and 0 on $K_{2\epsilon_1}(\Sigma)$. Then $\alpha' = \alpha - D_u g$ is supported in $K_{\epsilon_1}(\Sigma)$.

By Corollary 6.14, we get a finite dimensional subspace E' of $L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes u^*TX)$ such that $\mathrm{Im}\, D_u \oplus E' = L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes u^*TX)$. We may assume that E' consists of smooth sections since any section in $L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes u^*TX)$ can be approximated by sections in $C^\infty(\Sigma, \Lambda^{0,1}\Sigma \otimes u^*TX)$. The above claim shows that we may assume that all sections in E' have compact support in $K_{\epsilon_1}(\Sigma)$. Let E_ρ be the smallest $\mathrm{Aut}\,\rho$-invariant complex subspace which contains E'. Then E_ρ satisfies (1)–(5). □

Let $F_\rho = L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes u^*TX) \cap E_\rho^\perp$, where $E_\rho^\perp$ is the orthogonal complement of E_ρ in $L^2(\Sigma, \Lambda^{0,1}\Sigma \otimes u^*TX)$. Then F_ρ is a closed subspace of $L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes u^*TX)$, thus a Banach subspace of $L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes u^*TX)$. $F_\rho \cong L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes u^*TX)/E_\rho$. Let

$$\pi : L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes u^*TX) \to F_\rho$$

be the L^2-orthogonal projection. By (1) $\pi \circ D_u$ is surjective. Multiplication by i preserves the L^2 inner product, and (4) implies that F_ρ is a complex vector space. The action of $\mathrm{Aut}\,\rho$ also preserves the L^2 inner product, thus (5) implies that $\mathrm{Aut}\,\rho$ acts on F_ρ, and $\pi \circ D_u : W^{1,p}(\Sigma, \partial\Sigma, u^*TX, (u|_{\partial\Sigma})^*TL) \to F_\rho$ is $\mathrm{Aut}\,\rho$-equivariant.

Set $H_{\rho,\mathrm{map}} = \mathrm{Ker}(\pi \circ D_u)$. We have

$$\dim H_{\rho,\mathrm{map}} = \mu + N(1 - \tilde{g}) + \dim E_\rho,$$

where $\mu = \mu(u^*TX, (u|_{\partial\Sigma})^*TL)$, $2N = \dim_{\mathbb{R}} X$, and $\tilde{g}$ is the arithmetic genus of $\Sigma_{\mathbb{C}}$ as before.

The infinitesimal deformation of the domain is given by

$$H_{\rho,\mathrm{aut}} = \bigoplus_{i=1}^{\nu} U_i \oplus \bigoplus_{i'=1}^{\nu'} \hat{U}_{i'},$$

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where

$$\begin{aligned} U_i &= H^0(\hat{C}_i, T_{\hat{C}_i}(-x_1^i - \cdots - x_{\tilde{n}_i}^i)) \\ \hat{U}_{i'} &= H^0(\hat{\Sigma}_{i'}, \partial\hat{\Sigma}_{i'}, T_{\hat{\Sigma}_{i'}}(-p_1^{i'} - \cdots - p_{n_{i'}}^{i'}), T_{\partial\hat{\Sigma}_{i'}}(-q_1^{i'} - \ldots - q_{m_{i'}}^{i'})) \end{aligned}$$

$U_i = 0$ if and only $(\hat{C}_i, x_1^i, \cdots, x_{\tilde{n}_i}^i)$ is stable, and $\hat{U}_{i'} = 0$ if and only if $(\hat{\Sigma}_{i'}, (p_1^{i'}, \cdots, p_{n_{i'}}^{i'}), (q_1^{i'}, \ldots, q_{m_{i'}}^{i'}))$ is stable.

Put $\lambda = (\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h)$ as before. Then $H_{\rho,\mathrm{aut}}$ is the tangent space to $\mathrm{Aut}\,\lambda$ at the identity map. u is nonconstant on unstable components, so $\phi \in \mathrm{Aut}\,\lambda \mapsto u \circ \phi^{-1}$ induces an inclusion of vector spaces $H_{\rho,\mathrm{aut}} \subset H_{\rho,\mathrm{map}}$. Let $H'_{\rho,\mathrm{map}}$ be the L^2-orthogonal complement of $H_{\rho,\mathrm{aut}}$ in $H_{\rho,\mathrm{map}}$. Set

$$\begin{aligned} H_\rho &= H_{\rho,\mathrm{domain}} \times H_{\rho,\mathrm{map}}, \\ H'_\rho &= H_{\rho,\mathrm{domain}} \times H'_{\rho,\mathrm{map}}. \end{aligned}$$

With the above definitions, we are ready to state the main theorem of this section.

Theorem 6.19. *Let $\overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$ be equipped with the C^∞ topology. There are a neighborhood V'_ρ of 0 in H'_ρ such that $\mathrm{Aut}\,\rho$ acts on V'_ρ, an $\mathrm{Aut}\,\rho$-equivariant map $s_\rho : V'_\rho \to E_\rho$ such that $s_\rho(0) = 0$, and a continuous map $\psi_\rho : s_\rho^{-1}(0) \to \overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$ such that $s_\rho^{-1}(0)/\mathrm{Aut}\,\rho \to \overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$ gives a homeomorphism onto a neighborhood of ρ in $\overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$.*

$(V'_\rho, E_\rho, \mathrm{Aut}\,\rho, \psi_\rho, s_\rho)$ is a Kuranishi neighborhood of ρ.

6.4.1. *Pregluing: construction of approximate solutions.* In this section, we will modify u near nodes to obtain approximate J-holomorphic maps

$$u_{\eta,\eta'} : (\Sigma_{(0,\eta,\eta')}, \partial\Sigma_{(0,\eta,\eta')}) \to (X, L),$$

where the notation $\Sigma_{(\xi,\eta,\eta')}$ was introduced in Section 6.3.1, 6.3.2, 6.3.3 and will be used repeatedly in the rest of Section 6.4.

We first consider a neighborhood of an interior node. We will follow the construction in [31, Appendix A] closely. Recall that

$$B_{\epsilon_1,t} = \{(x,y) \in \mathbb{C}^2 \mid xy = t, |x| < \epsilon_1, |y| < \epsilon_1\}.$$

When $t \neq 0$, we have

$$B_{\epsilon_1,t} = \left\{ \left(x, \frac{t}{x}\right) \middle| x \in \mathbb{C}, \frac{r^2}{\epsilon_1} < |x| < \epsilon_1 \right\},$$

where $|t| = r^2$. We assume that $r < \frac{\epsilon_1^2}{4}$.

Let $B_{\epsilon_1} = \{x \in \mathbb{C} \mid |x| < \epsilon_1\}$. Two nonconstant J-holomorphic maps $f, g : B_{\epsilon_1} \to X$ such that $f(0) = g(0) = p$ determine a stable map $u : B_{\epsilon_1,0} \to X$ defined by $u(x,0) = f(x)$, $u(0,y) = g(y)$.

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Suppose that the images of f, g are contained in the geodesic ball $B_{r_1}(x)$ with respect to g_1, where r_1 is the injectivity radius of (M, g_1). We define $u_t : B_{\epsilon_1,t} \to X$ by

$$u_t\left(z, \frac{t}{z}\right) = \exp_p\left(\chi_1\left(\frac{z}{\sqrt{r}}\right)(\exp_p)^{-1}(f(z)) + \chi_1\left(\frac{r\sqrt{r}}{z}\right)(\exp_p)^{-1}(g\left(\frac{t}{z}\right))\right),$$

where $\chi_1 : \mathbb{C} \to [0,1]$ is a smooth cutoff function such that

$$\begin{aligned} \chi_1(z) &= \begin{cases} 1, & \text{if } |z| \geq 2 \\ 0, & \text{if } |z| \leq 1 \end{cases} \\ |\nabla \chi_1| &\leq 2 \end{aligned}$$

Then

$$u_t\left(z, \frac{t}{z}\right) = \begin{cases} u\left(0, \frac{t}{z}\right) = g(\frac{t}{z}) & \text{if } \frac{r^2}{\epsilon_1} < |z| < \frac{r\sqrt{r}}{2}, \\ p & \text{if } r\sqrt{r} \leq |z| \leq \sqrt{r}, \\ u(z,0) = f(z) & \text{if } 2\sqrt{r} < |z| < \epsilon_1. \end{cases}$$

Define $f_t, g_t : B_{\epsilon_1} \to X$ by

$$f_t(x) = \exp_p\left(\chi_1\left(\frac{x}{\sqrt{r}}\right)(\exp_p)^{-1}(f(x))\right)$$
$$g_t(y) = \exp_p\left(\chi_1\left(\frac{y}{\sqrt{r}}\right)(\exp_p)^{-1}(g(y))\right)$$

Then

$$f_t(x) = \begin{cases} f(x) & \text{if } 2\sqrt{r} < |x| < \epsilon_1 \\ p & \text{if } |x| < \sqrt{r} \end{cases}$$
$$g_t(y) = \begin{cases} g(y) & \text{if } 2\sqrt{r} < |x| < \epsilon_1 \\ p & \text{if } |x| < \sqrt{r} \end{cases}$$

f_t, g_t determine a map $v_t : B_{\delta_1,0} \to X$ defined by $v_t(x,0) = f_t(x)$, $v_t(0,y) = g_t(y)$. Define $F_t, F : B_{\epsilon_1} \to X$ by $f_t(x) = \exp_p(F_t(x))$, $f(x) = \exp_p(F(x))$. Then $F_t(0) = F(0) = 0$, and

$$F_t(x) = \chi_1\left(\frac{x}{\sqrt{r}}\right)F(x)$$

Lemma 6.20.

$$\| F_t - F \|_{W^{1,p}(B_{\epsilon_1})} \leq C \max_{\bar{B}_{2\sqrt{r}}} |\nabla F| r^{\frac{1}{p}},$$

where C is a universal constant, and $\bar{B}_{2\sqrt{r}} = \{x \in \mathbb{C} \mid |x| \leq 2\sqrt{r}\}$.

Proof. $F_t(x) - F(x) = 0$ for $|x| \geq 2\sqrt{r}$, and for $|x| < 2\sqrt{r}$,

$$\begin{aligned} |F_t(x) - F(x)| &= |(\chi_1\left(\frac{x}{\sqrt{r}}\right) - 1)F(x)| \leq |F(x)| \leq (\max_{\bar{B}_{2\sqrt{r}}} |\nabla F|)|x| \\ |\nabla(F_t(x) - F(x))| &= \left|\frac{1}{\sqrt{r}}\nabla\chi_1\left(\frac{x}{\sqrt{r}}\right)F(x) + \chi_1\left(\frac{x}{\sqrt{r}}\right)\nabla F(x)\right| \\ &\leq \frac{2}{\sqrt{r}}|F(x)| + |\nabla F(x)| \leq 5\max_{\bar{B}_{2\sqrt{r}}} |\nabla F| \end{aligned}$$

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The desired estimate is obtained by integrating over $B_{2\sqrt{r}}$, and noting that $r < 1$. $\square$

We embed (X, g_1) isometrically in $\mathbb{R}^l$ for some large l. Maps to X can be viewed as maps to $\mathbb{R}^l$, so we may subtract one map from another, and define their $L^{1,p}$ norms.

Corollary 6.21.

$$\| v_t - u \|_{W^{1,p}(B_{\epsilon_1})} \leq C(\max_{\bar{B}_{2\sqrt{r}}} |\nabla u|) r^{\frac{1}{p}},$$

where C depends on the C^∞ norms of $\exp|_{B_{r_1}(N_{L/X})}$ *and its inverse.*

Lemma 6.22.

$$\| \bar{\partial}_J u_t \|_{L^p} \leq C r^{\frac{1}{p}}$$

where C depends on the C^1 norm of u, the C^1 norm of J, the C^∞ norms of $\exp|_{B_{r_1}(N_{L/X})}$ *and its inverse.*

Proof. For $r \leq |z| < \epsilon_1$, we have

$$u_t\left(z, \frac{t}{z}\right) = f_t(z),$$

thus $\bar{\partial}_J u_t(z) = 0$ for $|z| > \sqrt{r}$. For $r \leq |z| \leq \sqrt{r}$, we have

$$\bar{\partial}_J u_t\left(z, \frac{t}{z}\right) = \bar{\partial}_J f_t(z) = \bar{\partial}_J (f_t - f)(z).$$

Let $z = x + iy$, then

$$\begin{aligned}
&2\bar{\partial}_J(f_t - f)\left(\frac{\partial}{\partial x}\right) \\
&= \frac{\partial f_t}{\partial x} + J(f_t)\frac{\partial f_t}{\partial y} - \frac{\partial f}{\partial x} - J(f)\frac{\partial f}{\partial y} \\
&= \frac{\partial}{\partial x}(f_t - f) + (J(f_t) - J(f))\frac{\partial f}{\partial y} + J(f)\frac{\partial}{\partial y}(f_t - f) + (J(f_t) - J(f))\frac{\partial}{\partial y}(f_t - f) \\
&\quad \left|\bar{\partial}_J(f_t - f)\left(\frac{\partial}{\partial x}\right)\right| \\
&\leq C_1(1 + \sup|J| + \sup|\nabla J||f_t - f|)|\nabla(f_t - f)| + C_2 \sup|\nabla J||f_t - f||\nabla f| \\
&\leq C_3(1 + \sup|J| + \sup|\nabla J|\sup|\nabla f||z|)|\nabla(f_t - f)| + C_4 \sup|\nabla J|\sup|\nabla f||z|\sup|\nabla f| \\
&\leq C_5|\nabla(f_t - f)| + C_6|z|
\end{aligned}$$

Similarly,

$$\left|\bar{\partial}_J(f_t - f)\left(\frac{\partial}{\partial y}\right)\right| \leq C_5|\nabla(f_t - f)| + C_6|z|$$

The case $\frac{r^2}{\epsilon_1} < |z| < r$ can be estimated similarly. Therefore,

$$\| \bar{\partial}_J u_t \|_{L^p} \leq C_7(\| \nabla(v_t - u) \|_{L^p} + \sqrt{r} r^{\frac{1}{p}}) \leq C r^{\frac{1}{p}}$$

where C depends on the C^1 norms of u and of J, C^∞ norms of $\exp^1|_{B_{r_1}(N_{L/X})}$ and its inverse. $\square$

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We next consider boundary nodes. For $t \in [0,\infty)$, define

$$B^+_{\epsilon_1,t} = \{(x,y) \in \mathbb{C}^2 \mid xy = t, |x| < \epsilon_1, |y| < \epsilon_1, \mathrm{Im} x \geq 0, \mathrm{Im} y \leq 0\}.$$

Then for $t = r^2 > 0$,

$$B^+_{\epsilon_1,t} = \left\{ \left(x, \frac{r^2}{x}\right) \middle| \, x \in \mathbb{C}, \frac{r^2}{\epsilon_1} < |x| < \epsilon_1, \mathrm{Im} x \geq 0 \right\}$$

Let $B^+_{\epsilon_1} = \{x \in \mathbb{C} \mid |x| < \epsilon_1, \mathrm{Im}\, x \geq 0\}$. Let $I_{\epsilon_1} = B^+_{\epsilon_1} \cap \mathbb{R}$, and $I_{\epsilon_1,t} = B^+_{\epsilon_1,t} \cap \mathbb{R} \times \mathbb{R}$. Two nonconstant J-holomorphic maps $f, g : (B^+_{\epsilon_1}, I_{\epsilon_1}) \to (X, L)$ such that $f(0) = g(0) = p$ determine a stable map $u : (B^+_{\epsilon_1,0}, I_{\epsilon_1,0}) \to (X, L)$ defined by $u(x, 0) = f(x)$, $u(0, y) = g(y)$. One can construct $u_t : (B^+_{\epsilon_1,t}, I_{\epsilon_1,t}) \to (X, L)$ and $v_t : (B^+_{\epsilon_1,0}, I_{\epsilon_1,0}) \to (X, L)$ as before.

Applying the above construction to each node, we obtain

$$\begin{aligned} u_{\eta,\eta'} &: (\Sigma_{(0,\eta,\eta')}, \partial\Sigma_{(0,\eta,\eta')}) \to (X, L) \\ v_{\eta,\eta'} &: (\Sigma, \partial\Sigma) \to (X, L) \end{aligned}$$

Lemma 6.22 implies

Lemma 6.23.

$$\| \bar{\partial}_J u_{\eta,\eta'} \|_{L^p} \leq C(|\eta| + |\eta'|)^{\frac{1}{2p}}$$

where C depends on the C^1 norms of u and J, C^∞ norms of $\exp|_{B_{r_1}}(N_{L/X})$ and its inverse.

The linearization of $\bar{\partial}_{J,\Sigma}$ at the stable $W^{1,p}$ map $v_{\eta,\eta'}$ is

$$D_{v_{\eta,\eta'}} : W^{1,p}(\Sigma, \partial\Sigma, v^*_{\eta,\eta'}TX, (v_{\eta,\eta'}|_{\partial\Sigma})^*TL) \to L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes v^*_{\eta,\eta'}TX).$$

Lemma 6.24.

$$\lim_{(\eta,\eta')\to 0} \| D_{v_{\eta,\eta'}} \| = \| D_u \|$$

Proof. We have a bundle isomorphism

$$P_0 : (u^*TX, (u|_{\partial\Sigma})^*TL) \to (v^*_{\eta,\eta'}TX, (v_{\eta,\eta'}|_{\partial\Sigma})^*TL)$$

given by parallel transport along the unique length minimizing geodesic from $u(z)$ to $v_{\eta,\eta'}(z)$. This also gives

$$P_1 : \Lambda^1\Sigma \otimes u^*TX \to \Lambda^1\Sigma \otimes v^*_{\eta,\eta'}TX$$

and

$$P_1' = \pi \circ P_1 \circ i : \Lambda^{0,1}\Sigma \otimes u^*TX \to \Lambda^{0,1}\Sigma \otimes v^*_{\eta,\eta'}TX,$$

where $i : \Lambda^{0,1}\Sigma \otimes u^*TX \to \Lambda^1\Sigma \otimes u^*TX$ is the inclusion, and $\pi : \Lambda^1\Sigma \otimes v^*_{\eta,\eta'}TX \to \Lambda^{0,1}\Sigma \otimes v^*_{\eta,\eta'}TX$ is the projection. P_0, P_1' induces

$$\begin{aligned} \tilde{P}^{-1}_{\eta,\eta',0} &: W^{1,p}(\Sigma, \partial\Sigma, v^*_{\eta,\eta'}TX, (v_{\eta,\eta'}|_{\partial\Sigma})^*TL) \to W^{1,p}(\Sigma, \partial\Sigma, u^*TX, (u|_{\partial\Sigma})^*TL) \\ \tilde{P}_{\eta,\eta',1} &: L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes u^*TX) \to L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes v^*_{\eta,\eta'}TX). \end{aligned}$$

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Define

$$D'_{\eta,\eta'} = \tilde{P}_{\eta,\eta',1} \circ D_u \circ \tilde{P}^{-1}_{\eta,\eta',0} :$$
$$W^{1,p}(\Sigma, \partial\Sigma, v^*_{\eta,\eta'}TX, (v_{\eta,\eta'}|_{\partial\Sigma})^*TL) \to L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes v^*_{\eta,\eta'}TX),$$

then

$$\lim_{(\eta,\eta')\to(0,0)} \| D'_{\eta,\eta'} \| = \| D_u \| .$$

From Lemma 6.12, we see that

$$\begin{aligned} \| (D_{v_{\eta,\eta'}} - D'_{\eta,\eta'})w \| &\leq C_2 \| u - v_{\eta,\eta'} \|_{C^0} \| \nabla w \|_{L^p} + \| w \|_{C^0} \| du - dv_{\eta,\eta'} \|_{L^p} \\ &\leq C_3 \| u - v_{\eta,\eta'} \|_{W^{1,p}} \| w \|_{W^{1,p}} \\ \| D_{v_{\eta,\eta'}} - D'_{\eta,\eta'} \| &\leq C_3 \| u - v_{\eta,\eta'} \|_{W^{1,p}} \end{aligned}$$

which tends to 0 as $(\eta, \eta') \to (0, 0)$. □

6.4.2. *Gluing: construction of exact solutions.* The goal of this section is to construct a local parametrization of solutions to $\pi\circ\bar{\partial}_J v = 0$ near the approximate J-holomorphic map $u_{\eta,\eta'}$ constructed in Section 6.4.1. The main result in this Section is Proposition 6.32.

Let $B_{\delta_2} \times D(d_1, \ldots, d_{l_0}) \times D'(d'_1, \ldots, d'_{l_1})$ be the neighborhood of the origin in $H_{\rho,\text{domain}}$ as in Section 6.3.3. We write D_d for $D(d_1, \ldots, d_{l_0})$, and $D'_{d'}$ for $D'(d'_1, \ldots, d'_{l_1})$, as in Section 6.3.4. We have seen that there is a family of prestable bordered Riemann surfaces

$$(\pi : \mathcal{C} \to B_{\delta_2} \times D_d \times D'_{d'}; \mathbf{s}; \mathbf{t}^1, \ldots, \mathbf{t}^h)$$

of type (g, h) with (n, m) marked points, together with a family of admissible metrics, such that $\pi^{-1}(0) = \Sigma$. There is a map $\mathcal{C} \to \Sigma$ whose restriction to each fiber of π is a smooth strong deformation $\kappa_{(\xi,\eta,\eta')} : \lambda_{(\xi,\eta,\eta')} \to \lambda$.

Let B be the image of the map $i : B_{\delta_2} \times D_d \times D'_{d'} \to \widetilde{M}_{(g,h),(n,\vec{m})}$ given by $(\xi, \eta, \eta') \mapsto \lambda_{(\xi,\eta,\eta')}$. Then B is a neighborhood of λ in the C^∞ topology. Using the family of admissible metrics, we define $W_B = \cup_{\lambda'\in B} W^{1,p}_{\lambda'}$. Let $M_B = \overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu) \cap W_B$. Then $\rho \in M_B$.

We have a Cartesian diagram

$$\begin{array}{ccc} W_{\tilde{B}} & \longrightarrow & W_B \\ {\scriptstyle\tilde{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ B_{\delta_2} \times D_d \times D'_{d'} = \tilde{B} & \xrightarrow{i} & B \end{array}$$

Let $S : \tilde{B} \to W_{\tilde{B}}$ be given by $(\xi, \eta, \eta') \mapsto u_{\eta,\eta'}$.

We first extend $E_\rho \subset L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes u^*TX)$ to a trivial bundle over $S(\{0\} \times D_d \times D'_{d'})$. Recall that elements in E_ρ are supported on $K_{\epsilon_1}(\Sigma)$. Since $v_{\eta,\eta'} = u$ on $K_{\epsilon_1}(\Sigma)$, E_ρ can be viewed as a subspace $E_{\eta,\eta'}$ of $L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes v^*_{\eta,\eta'}TX)$. Let $F_{\eta,\eta'}$ be the L^2-orthogonal complement of $E_{\eta,\eta'}$ in $L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes v^*_{\eta,\eta'}TX)$, and let

$$\pi_{\eta,\eta'} : L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes v^*_{\eta,\eta'}TX) \to F_{\eta,\eta'}$$

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be the L^2-orthogonal projection. Then for sufficiently small η, η',

$$\pi_{\eta,\eta'} \circ D_{v_{\eta,\eta'}} : W^{1,p}(\Sigma, \partial\Sigma, v_{\eta,\eta'}^* TX, (v_{\eta,\eta'}|_{\partial\Sigma})^* TL) \to F_{\eta,\eta'}$$

is a surjection. Let $H_{(\eta,\eta')} = \mathrm{Ker}(\pi_{\eta,\eta'} \circ D_{v_{\eta,\eta'}})$, and let $H_{(\eta,\eta')}^{\perp}$ be its L^2-orthogonal complement in $W^{1,p}(\Sigma, \partial\Sigma, v_{\eta,\eta'}^* TX, (v_{\eta,\eta'}|_{\partial\Sigma})^* TL)$. Then we have an isomorphism

$$\pi_{\eta,\eta'} \circ D_{v_{\eta,\eta'}} : H_{\eta,\eta'}^{\perp} \to F_{\eta,\eta'}$$

whose inverse is

$$\hat{Q}_{\eta,\eta'} : F_{\eta,\eta'} \to H_{\eta,\eta'}^{\perp}.$$

We have

$$Q_{\eta,\eta'} = i \circ \hat{Q}_{\eta,\eta'} : F_{\eta,\eta'} \to W^{1,p}(\Sigma, \partial\Sigma, v_{\eta,\eta'}^* TX, (v_{\eta,\eta'}|_{\partial\Sigma})^* TL)$$

where

$$i : H_{(\eta,\eta')}^{\perp} \to W^{1,p}(\Sigma, \partial\Sigma, v_{\eta,\eta'}^* TX, (v_{\eta,\eta'}|_{\partial\Sigma})^* TL)$$

is the inclusion. $Q_{\eta,\eta'}$ is a right inverse of $\pi_{\eta,\eta'} \circ D_{v_{\eta,\eta'}}$.

By Lemma 6.24, we may choose d, d' sufficiently small such that $\hat{Q}_{\eta,\eta'}$ exists and $\| Q_{\eta,\eta'} \| \leq M$ for all $(\eta, \eta') \in D_d \times D'_{d'}$, where M is a constant.

We now extend $E \to S(\{0\} \times D_d \times D'_{d'})$ to a neighborhood $U_{\tilde{B}}$ of $S(\tilde{B})$ in $W_{\tilde{B}}$. Put $\rho' = (\lambda_{(\xi,\eta,\eta')}, f)$, where $f : (\Sigma_{(\xi,\eta,\eta')}, \partial\Sigma_{(\xi,\eta,\eta')}) \to (X, L)$ is a stable $W^{1,p}$ map such that

$$\sup_{\Sigma_{(\xi,\eta,\eta')}} d_1(u_{\eta,\eta'}(z), f(z))$$

is less than the injectivity radius of g_1, where d_1 is the geodesic distance of g_1.

We have a bundle isomorphism

$$P_0 : (u_{\eta,\eta'}^* TX, (u_{\eta,\eta}|_{\partial\Sigma_{(\xi,\eta,\eta')}})^* TL) \to (f^* TX, (f|_{\partial\Sigma_{(\xi,\eta,\eta')}})^* TL)$$

given by the parallel transport along the unique length minimizing geodesic from $u_{\eta,\eta'}(z)$ to $f(z)$, which gives

$$\begin{aligned} P_1 : \quad & \Lambda^1\Sigma_{(\xi,\eta,\eta')} \otimes u_{\eta,\eta'}^* TX \to \Lambda^1\Sigma_{(\xi,\eta,\eta')} \otimes f^* TX \\ P_1' = \pi \circ P_1 \circ i : \quad & \Lambda^{0,1}\Sigma_{(\xi,\eta,\eta')} \otimes u_{\eta,\eta'}^* TX \to \Lambda^{0,1}\Sigma_{(\xi,\eta,\eta')} \otimes f^* TX \end{aligned}$$

where $i : \Lambda^{0,1}\Sigma_{(\xi,\eta,\eta')} \otimes u_{\eta,\eta'}^* TX \to \Lambda^1\Sigma_{(\xi,\eta,\eta')} \otimes u_{\eta,\eta'}^* TX$ is the inclusion, and $\Lambda^1\Sigma_{(\xi,\eta,\eta')} \otimes f^* TX \to \Lambda^{0,1}\Sigma_{(\xi,\eta,\eta')} \otimes f^* TX$ is the projection. We have

$$\tilde{P} : L^p(\Sigma_{(\xi,\eta,\eta')}, \Lambda^{0,1}\Sigma_{(\xi,\eta,\eta')} \otimes u_{\eta,\eta'}^* TX) \to L^p(\Sigma_{(\xi,\eta,\eta')}, \Lambda^{0,1}\Sigma_{(\xi,\eta,\eta')} \otimes f^* TX).$$

Let $E_{\rho'} = \tilde{P} E_\rho \cong E_\rho$. Then we have a trivial bundle $E \to U_{\tilde{B}}$ together with a trivialization $\Phi : E \cong U_{\tilde{B}} \times E_\rho$.

Let $F_{\rho'}$ be the L^2-orthogonal complement of $E_{\rho'}$ in $L^p(\Sigma_{(\xi,\eta,\eta')}, \Lambda^{0,1}\Sigma_{(\xi,\eta,\eta')} \otimes f^* TX)$, and let

$$\pi_{\rho'} : L^p(\Sigma_{(\xi,\eta,\eta')}, \Lambda^{0,1}\Sigma_{(\xi,\eta,\eta')} \otimes f^* TX) \to F_{\rho'}$$

be the L^2-orthogonal projection.

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We will use $Q_{\eta,\eta'}$, the right inverse of $\pi_{\eta,\eta'} \circ D_{v_{\eta,\eta'}}$, to construct an approximate right inverse $Q'_{(\xi,\eta,\eta')}$ of $\pi_{(\xi,\eta,\eta')} \circ D_{(\xi,\eta,\eta'),u_{\eta,\eta'}}$, where $\pi_{(\xi,\eta,\eta')} = \pi_{(\lambda_{(\xi,\eta,\eta')},u_{\eta,\eta'})}$, and $D_{(\xi,\eta,\eta'),u_{\eta,\eta'}}$ is the linearization of $\bar\partial_{J,\Sigma_{(\xi,\eta,\eta')}}$ at $u_{\eta,\eta'}$. We will use the cutoff function constructed in [31, A.1]. The construction in this section works for $W^{1,p}$ but not for general $W^{k,p}$.

Lemma 6.25. *For any $r \in (0,1)$, there is a smooth cutoff function $\chi_r : \mathbb{C} \to [0,1]$ such that*

$$\chi_r(z) = \begin{cases} 1 & \text{if } |z| \le r\sqrt{r} \\ 0 & \text{if } |z| \ge r \end{cases}$$

$$\int_{|z|\le r} \chi_r \le \frac{4\pi}{|\log r|}.$$

Proof. We will follow the proof of [31, Lemma A.1.1]. We first define a cutoff function of class $W^{1,2}$ by

$$\beta_r(z) = \begin{cases} 1 & \text{for } |z| \le r\sqrt{r} \\ 2\left(\frac{\log|z|}{\log r} - 1\right) & \text{for } r\sqrt{r} \le |z| \le r \\ 0 & \text{for } |z| \ge r \end{cases}$$

Then we have

$$|\nabla \beta_r(z)| = \frac{2}{|z||\log r|}$$

for $r\sqrt{r} \le |z| \le r$, so

$$\int_{r\sqrt{r}\le|z|\le r} |\nabla\beta_r(z)|^2 = \frac{4\pi}{|\log r|}.$$

To obtain a smooth function χ_r, take the convolution with $\phi_N(z) = N^2\phi(Nz)$ where N is large and $\phi : \mathbb{C} \to \mathbb{R}$ is a smooth function with support in the unit ball and mean value 1. □

Let $p > 2$ be fixed as before.

Lemma 6.26. *For every $r \in (0,1)$, there exists a smooth cutoff function $\chi_r : \mathbb{C} \to [0,1]$ as in Lemma 6.25 such that*

$$\| (\nabla\chi_r)w \|_{L^p} \le C \| w \|_{W^{1,p}} |\log r|^{\frac{1}{p}-1}$$

for any $w \in W^{1,p}(\mathbb{C})$ with $w(0) = 0$.

Proof. This follows from the proof of [31, Lemma A.1.2]. □

We now look at the local model of an interior node. Let $u : B_{\epsilon_1,0} \to X$ be a stable map, and construct smooth maps $u_t : B_{\epsilon_1,t} \to X$, $v_t : B_{\epsilon_1,0} \to X$ as before. We now define linear maps

$$e_t : L^p(B_{\epsilon_1,t}, \Lambda^{0,1}B_{\epsilon_1,t} \otimes u_t^*TX) \to L^p(B_{\epsilon_1,0}, \Lambda^{0,1}B_{\epsilon_1,0} \otimes v_t^*TX)$$
$$g_t : W^{1,p}(B_{\epsilon_1,0}, v_t^*TX) \to W^{1,p}(B_{\epsilon_1,t}, u_t^*TX)$$

Given $s \in L^p(B_{\epsilon_1,t}, \Lambda^{0,1}B_{\epsilon_1,t} \otimes u_t^*TX)$, we define

$$e_t(s) \in L^p(B_{\epsilon_1,0}, \Lambda^{0,1}B_{\epsilon_1,0} \otimes v_t^*TX)$$

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by

$$e_t(s)(x,0) = \begin{cases} s(x,\frac{t}{x}) & \text{if } r \le |x| \le \epsilon_1 \\ 0 & \text{if } |x| < r \end{cases}$$

$$e_t(s)(0,y) = \begin{cases} s(\frac{t}{y},y) & \text{if } r \le |y| \le \epsilon_1 \\ 0 & \text{if } |y| < r \end{cases}$$

The above definition is valid since

$$u_t\left(z,\frac{t}{z}\right) = \begin{cases} v_t(0,\frac{t}{z}) & \text{if } \frac{r^2}{\epsilon_1} \le |z| \le r \\ v_t(z,0) & \text{if } r \le |x| \le \epsilon_1 \end{cases}$$

Given $w \in W^{1,p}(B_{\epsilon_1,0}, v_t^*TX)$, we define

$$g_t(w) \in W^{1,p}(B_{\epsilon_1,t}, u_t^*TX)$$

by

$$g_t(w)\left(z,\frac{t}{z}\right) = \begin{cases} w(z,0) & \text{if } \sqrt{r} \le |z| \le \epsilon_1 \\ w(z,0) + (1-\chi_r(\frac{t}{z}))(w(0,\frac{t}{z}) - w(0,0)) & \text{if } r \le |z| \le \sqrt{r} \\ w(0,\frac{t}{z}) + (1-\chi_r(z))(w(z,0) - w(0,0)) & \text{if } r\sqrt{r} \le |z| \le r \\ w(0,\frac{t}{z}) & \text{if } \frac{r^2}{\epsilon_1} \le |z| \le r\sqrt{r} \end{cases}$$

Lemma 6.27. *If* $s \in L^p(B_{\epsilon_1,t}, \Lambda^{0,1}B_{\epsilon_1,t} \otimes u_t^*TX)$, $w \in W^{1,p}(B_{\epsilon_1,0}, v_t^*TX)$ *satisfy* $D_{v_t}w = e_t(s)$, *then*

$$\| D_{u_t} g_t(w) - s \|_{L^p} \le C \| w \|_{W^{1,p}} (|\log r|^{\frac{1}{p}-1} + \| u \|_{W^{1,p}} r^{\frac{1}{p}})$$

Proof. We have

$$D_{u_t} \circ g_t(w)\left(z,\frac{t}{z}\right) = \begin{cases} D_{v_t}w(z,0) = e_t(s)(z,0) = s(z,\frac{t}{z}) & \text{for } \sqrt{r} \le |z| \le \epsilon_1, \\ D_{v_t}w(0,\frac{t}{z}) = e_t(s)(0,\frac{t}{z}) = s(z,\frac{t}{z}) & \text{for } \frac{r^2}{\epsilon_1} \le |z| \le r\sqrt{r}. \end{cases}$$

We first consider the case $r \le |z| \le \sqrt{r}$. We have

$$\begin{aligned} & D_{u_t} \circ g_t(w)\left(z,\frac{t}{z}\right) \\ = \; & D_{v_t}w(z,0) + D(1-\chi_r\left(\frac{t}{z}\right))(w(0,\frac{t}{z}) - w(0,0)) \\ & +(1-\chi_r\left(\frac{t}{z}\right))(D_{v_t}w(0,\frac{t}{z}) - D_{v_t}w(0,0)) \end{aligned}$$

where D is a derivation, thus

$$|D(1-\chi_r\left(\frac{t}{z}\right))| \le C_1 \left| \frac{t}{z^2}\nabla\chi_r\left(\frac{t}{z}\right)\right|.$$

We also have

$$\begin{aligned} D_{v_t}w(z,0) &= e_t(s)(z,0) = s(z,\frac{t}{z}), \\ D_{v_t}w(0,\frac{t}{z}) &= e_t(s)(0,\frac{t}{z}) = 0, \end{aligned}$$

and

$$\begin{aligned} |D_{v_t} w(0,0)| &\leq C_2 \parallel v_t \parallel_{W^{1,p}} |w(0,0)| \\ &\leq C_2 \parallel v_t \parallel_{W^{1,p}} \parallel w \parallel_{C^0} \\ &\leq C_3 \parallel v_t \parallel_{W^{1,p}} \parallel w \parallel_{W^{1,p}} \end{aligned}$$

Let h_0 be the metric on $A(r,\sqrt{r})$ given by $A(r,\sqrt{r}) \subset \mathbb{C}$, and h_1 be the metric on $A(r,\sqrt{r})$ given by the embedding $z \in A(r,\sqrt{r}) \mapsto (z, \frac{t}{z}) \in \mathbb{C}^2$. Then

$$h_1 = \left(1 + \left(\frac{r}{|z|}\right)^4\right) h_0,$$

so $h_0 \leq h_1 \leq 2h_0$. To estimate the $L^{1,p}$ norm defined by any metric which is an interpolation of h_0 and h_1, it suffices to calculate in h_0.

$$\begin{aligned} & \int_{r\leq|z|\leq\sqrt{r}} |D_{u_t} \circ g_t(w)(z, \frac{t}{z}) - s(z, \frac{t}{z})|^p \frac{i}{2} dz \wedge d\bar{z} \\ \leq\ & C_1^p \int_{\sqrt{r}\leq|z|\leq r} \left|\nabla \chi_r\left(\frac{t}{z}\right)\right|^p \left|w(0,\frac{t}{z}) - w(0,0)\right|^p \left(\frac{r}{|z|}\right)^{2p} \frac{i}{2} dz \wedge d\bar{z} \\ & + C_4 \parallel v_t \parallel^p_{W^{1,p}} \parallel w \parallel^p_{W^{1,p}} r \end{aligned}$$

where

$$\begin{aligned} & \int_{r\leq|z|\leq\sqrt{r}} \left|\nabla \chi_r\left(\frac{t}{z}\right)\right|^p |w(0,\frac{t}{z}) - w(0,0)|^p \left(\frac{r}{|z|}\right)^{2p} \frac{i}{2} dz \wedge d\bar{z} \\ =\ & \int_{r\sqrt{r}\leq|y|\leq r} |\nabla \chi_r(y)|^p |w(0,y) - w(0,0)|^p \left(\frac{|y|}{r}\right)^{2p-4} \frac{i}{2} dy \wedge d\bar{y} \\ =\ & \int_{r\sqrt{r}\leq|y|\leq r} |\nabla \chi_r(y)|^p |w(0,y) - w(0,0)|^p \frac{i}{2} dy \wedge d\bar{y} \\ \leq\ & C_5 \parallel w \parallel^p_{W^{1,p}} |\log r|^{1-p} \end{aligned}$$

We finally consider the case $r\sqrt{r} \leq |z| \leq r$. Let $y = \frac{t}{z}$, then $r \leq |y| \leq \sqrt{r}$, and

$$\begin{aligned} g_t(w)\left(z, \frac{t}{z}\right) &= g_t(w)\left(\frac{t}{y}, y\right) \\ &= w(0,y) + (1 - \chi_r\left(\frac{t}{y}\right))(w(\frac{t}{y}, 0) - w(0,0)), \end{aligned}$$

which is the same as the case $r \leq |z| \leq \sqrt{r}$. So we conclude that

$$\parallel D_{u_t} g_t(w) - s \parallel_{L^p} \leq C \parallel w \parallel_{W^{1,p}} (|\log r|^{\frac{1}{p}-1} + \parallel u \parallel_{W^{1,p}} r^{\frac{1}{p}})$$

since v_t converges to u in $W^{1,p}$ norm. □

We next look at the local model of a boundary node. Let $u : (B^+_{\epsilon_1,0}, I_{\epsilon_1,0}) \to (X,L)$ be a stable map, and construct smooth maps $u_t : (B^+_{\epsilon_1,t}, I_{\epsilon_1,t}) \to (X,L)$, $v_t : (B^+_{\epsilon_1,0}, I_{\epsilon_1,0}) \to (X,L)$ as before. We

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define linear maps

$$\begin{aligned}
e_t &: L^p(B^+_{\epsilon_1,t}, \Lambda^{0,1}B_{\epsilon_1,t} \otimes u_t^*TX) \to L^p(B^+_{\epsilon_1,0}, \Lambda^{0,1}B_{\epsilon_1,0} \otimes v_t^*TX) \\
g_t &: W^{1,p}(B^+_{\epsilon_1,0}, I_{\epsilon_1,0}, v_t^*TX, (v_t|_{I_{\epsilon_1,0}})^*TL) \to W^{1,p}(B^+_{\epsilon_1,t}, I_{\epsilon_1,t}, u_t^*TX, (u_t|_{I_{\epsilon_1,t}})^*TL)
\end{aligned}$$

in exactly the same way as for $B_{\epsilon_1,t}$, $B_{\epsilon_1,0}$. Then we have

Lemma 6.28. *If $s \in L^p(B^+_{\epsilon_1,t}, \Lambda^{0,1}B_{\epsilon_1,t} \otimes u_t^*TX)$, $w \in W^{1,p}(B^+_{\epsilon_1,0}, I_{\epsilon_1,0}, v_t^*TX, (v_t|_{I_{\epsilon_1,0}})^*TL)$ satisfy $D_{v_t}w = e_t(s)$, then*

$$\| D_{u_t}g_t(w) - s \|_{L^p} \leq C \| w \|_{W^{1,p}} (|\log r|^{\frac{1}{p}-1} + \| u \|_{W^{1,p}} r^{\frac{1}{p}}).$$

We now apply above construction to each node to obtain linear maps

$$\begin{aligned}
e_{\eta,\eta'} :\ & L^p(\Sigma_{(0,\eta,\eta')}, \Lambda^{0,1}\Sigma_{(0,\eta,\eta')} \otimes u^*_{\eta,\eta'}TX) \to L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes v^*_{\eta,\eta'}TX) \\
g_{\eta,\eta'} :\ & W^{1,p}(\Sigma, \partial\Sigma, v^*_{\eta,\eta'}TX, (v_{\eta,\eta'}|_{\partial\Sigma})^*TL) \\
& \to W^{1,p}(\Sigma_{(0,\eta,\eta')}, \partial\Sigma_{(0,\eta,\eta')}, u^*_{\eta,\eta'}TX, (u_{\eta,\eta'}|_{\partial\Sigma_{(0,\eta,\eta')}})^*TL)
\end{aligned}$$

Let

$$\begin{aligned}
Q'_{(0,\eta,\eta')} = g_{\eta,\eta'} \circ Q_{\eta,\eta'} \circ \pi_{\eta,\eta'} \circ \left(e_{\eta,\eta'}|_{F_{\eta,\eta'}} \right) :& \\
F_{\eta,\eta'} \to W^{1,p}(\Sigma_{(0,\eta,\eta')}, \partial\Sigma_{(0,\eta,\eta')}, u^*_{\eta,\eta'}TX, (u_{\eta,\eta'}|_{\partial\Sigma_{(0,\eta,\eta')}})^*TL).&
\end{aligned}$$

The operator norm of $Q'_{(0,\eta,\eta')}$ has a uniform bound independent of η, η' since the operator norm of $Q_{\eta,\eta'}$ has a uniform bound independent of η, η'. We now show that $Q'_{(0,\eta,\eta')}$ is an approximate right inverse of $\pi_{(0,\eta,\eta')} \circ D_{(0,\eta,\eta'),u_{\eta,\eta'}}$.

Proposition 6.29.

$$\| \left(\pi_{(0,\eta,\eta')} \circ D_{(0,\eta,\eta'),u_{\eta,\eta'}} \circ Q'_{(0,\eta,\eta')} \right) s - s \|_{L^p} \leq C(|\log(|\eta| + |\eta'|)|^{\frac{1}{p}-1}) \| s \|_{L^p}$$

where C depends on $\| u \|_{W^{1,p}}$.

Proof. Let $\rho(\eta,\eta') = (\lambda_{(0,\eta,\eta')}, u_{\eta,\eta'})$. Given $s \in F_{\rho(\eta,\eta')}$, let

$$\begin{aligned}
s_1 &= \pi_{\eta,\eta'} \circ e_{\eta,\eta'}(s) \in F_{\eta,\eta'}, \\
t_1 &= e_{\eta,\eta'}(s) - s_1 \in E_{\eta,\eta'}, \\
w &= Q_{\eta,\eta'}(s_1) \in W^{1,p}(\Sigma, \partial\Sigma, v^*_{\eta,\eta'}TX, (v_{\eta,\eta'}|_{\partial\Sigma})^*TL),
\end{aligned}$$

then

$$\pi_{\eta,\eta'} \circ D_{v_{\eta,\eta'}} w = s_1 = e_{\eta,\eta'}(s) - t_1.$$

thus $D_{v_{\eta,\eta'}} w = e_{\eta,\eta'}(s) + t_2$ for some $t_2 \in E_{\eta,\eta'}$. There is a unique $t \in E_{\rho(\eta,\eta')}$ such that $e_{\eta,\eta'}(t) = t_2$. We have

$$D_{v_{\eta,\eta'}} w = e_{\eta,\eta'}(s + t),$$

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hence by Lemma 6.27, 6.28,

$$\begin{aligned} & \| D_{(0,\eta,\eta'),u_{\eta,\eta'}} \circ g_{\eta,\eta'}(w) - (s+t) \|_{L^p\left(\kappa^{-1}_{\eta,\eta'}(N_{\epsilon_1}(\Sigma))\right)} \\ \leq \quad & C_1 \| w \|_{W^{1,p}} \left(|\log(|\eta| + |\eta'|)|^{\frac{1}{p}-1} + (|\eta| + |\eta'|)^{\frac{1}{2p}} \right) \end{aligned}$$

where C_1 depends on $\| u \|_{W^{1,p}}$.

We have

$$\pi_{(0,\eta,\eta')} \circ D_{(0,\eta,\eta'),u_{\eta,\eta'}} \circ Q'_{\eta,\eta'} s - s = \pi_{(0,\eta,\eta')} \left(D_{(0,\eta,\eta'),u_{\eta,\eta'}} \circ g_{\eta,\eta'}(w) - (s+t) \right)$$

$$\begin{aligned} \| w \|_{W^{1,p}} &= \| Q_{\eta,\eta'} \circ \pi_{\eta,\eta'} \circ e_{\eta,\eta'}(s) \|_{W^{1,p}} \\ &\leq C_2 \| \pi_{\eta,\eta'} \circ e_{\eta,\eta'}(s) \|_{L^p} \\ &\leq C_3 \| s \|_{L^p} \end{aligned}$$

Therefore,

$$\begin{aligned} & \| \pi_{(0,\eta,\eta')} \circ D_{(0,\eta,\eta'),u_{\eta,\eta'}} \circ Q'_{(0,\eta,\eta')} s - s \|_{L^p\left(\kappa^{-1}_{\eta,\eta'}(N_{\epsilon_1}(\Sigma))\right)} \\ \leq \quad & C_4 \left(|\log(|\eta| + |\eta'|)|^{\frac{1}{p}-1} + (|\eta| + |\eta'|)^{\frac{1}{2p}} \right) \| s \|_{L^p} \\ \leq \quad & C_5 |\log(|\eta| + |\eta'|)|^{\frac{1}{p}-1} \| s \|_{L^p} . \end{aligned}$$

□

Let

$$p_{(\xi,\eta,\eta')} : L^p(\Sigma_{(\xi,\eta,\eta')}, \Lambda^{0,1}\Sigma_{(\xi,\eta,\eta')} \otimes u^*_{\eta,\eta'} TX) \to L^p(\Sigma_{(\xi,\eta,\eta')}, \Lambda^{0,1}\Sigma_{(0,\eta,\eta')} \otimes u^*_{\eta,\eta'} TX)$$

be the map determined by the bundle isomorphism $P \circ i : \Lambda^{0,1}\Sigma_{(\xi,\eta,\eta')} \to \Lambda^{0,1}\Sigma_{(0,\eta,\eta')}$, where $i : \Lambda^{0,1}\Sigma_{(\xi,\eta,\eta')} \to \Lambda^{1}\Sigma_{(\xi,\eta,\eta')} \cong \Lambda^{1}\Sigma_{(0,\eta,\eta')}$ is the inclusion, and $P : \Lambda^{1}\Sigma_{(0,\eta,\eta')} \to \Lambda^{0,1}\Sigma_{(0,\eta,\eta')}$ is the projection. Let

$$\begin{aligned} Q'_{(\xi,\eta,\eta')} = g_{\eta,\eta'} \circ Q_{\eta,\eta'} \circ \pi_{\eta,\eta'} \circ e_{\eta,\eta'} \circ (p_{(\xi,\eta,\eta')}|_{F_{(\xi,\eta,\eta')}}) : \\ F_{(\xi,\eta,\eta')} \to W^{1,p}(\Sigma_{(\xi,\eta,\eta')}, \partial\Sigma_{(\xi,\eta,\eta')}, u^*_{\eta,\eta'} TX, (u_{\eta,\eta'}|_{\partial\Sigma_{(\xi,\eta,\eta')}})^* TL) \end{aligned}$$

where $F_{(\xi,\eta,\eta')} = F_{(\lambda_{(\xi,\eta,\eta')}, u_{\eta,\eta'})}$. We have

Proposition 6.30.

$$\| \left(\pi_{(\xi,\eta,\eta')} \circ D_{(\xi,\eta,\eta'),u_{\eta,\eta'}} \circ Q'_{(\xi,\eta,\eta')} \right) s - s \|_{L^p} \leq C(|\xi| + |\log(|\eta| + |\eta'|)|^{\frac{1}{p}-1}) \| s \|_{L^p}$$

Proof. We identify

$$W^{1,p}(\Sigma_{(\xi,\eta,\eta')}, \partial\Sigma_{(\xi,\eta,\eta')}, u^*_{\eta,\eta'} TX, (u_{\eta,\eta'}|_{\partial\Sigma_{(\xi,\eta,\eta')}})^* TL)$$

with

$$W^{1,p}(\Sigma_{(0,\eta,\eta')}, \partial\Sigma_{(0,\eta,\eta')}, u^*_{\eta,\eta'} TX, (u_{\eta,\eta'}|_{\partial\Sigma_{(0,\eta,\eta')}})^* TL),$$

and embed the spaces

$$L^p(\Sigma_{(\xi,\eta,\eta')}, \Lambda^{0,1}\Sigma_{(\xi,\eta,\eta')} \otimes u^*_{\eta,\eta'} TX)$$

and

$$L^p(\Sigma_{(0,\eta,\eta')}, \Lambda^{0,1}\Sigma_{(0,\eta,\eta')} \otimes u^*_{\eta,\eta'}TX)$$

into

$$L^p(\Sigma_{(0,\eta,\eta')}, \Lambda^{1}\Sigma_{(0,\eta,\eta')} \otimes u^*_{\eta,\eta'}TX).$$

We extend the domain of $Q'_{(0,\eta,\eta')}$ to $L^p(\Sigma_{(0,\eta,\eta')}, \Lambda^{0,1}\Sigma_{(0,\eta,\eta')} \otimes u^*_{\eta,\eta'}TX)$, and that of $Q'_{(\xi,\eta,\eta')}$ to $L^p(\Sigma_{(0,\eta,\eta')}, \Lambda^{0,1}\Sigma_{(0,\eta,\eta')} \otimes u^*_{\eta,\eta'}TX)$. In other words, we have

$$Q'_{(0,\eta,\eta')} = g_{\eta,\eta'} \circ Q_{\eta,\eta'} \circ \pi_{\eta,\eta'} \circ e_{\eta,\eta'}, \quad Q'_{(\xi,\eta,\eta')} = Q'_{(0,\eta,\eta')} \circ p_{(\xi,\eta,\eta')}.$$

With the above convention, we have

$$\begin{aligned} & \left(\pi_{(\xi,\eta,\eta')} \circ D_{(\xi,\eta,\eta'),u_{\eta,\eta'}} \circ Q'_{(\xi,\eta,\eta')}\right) s - s \\ = \;& (\pi_{(\xi,\eta,\eta')} \circ D_{(\xi,\eta,\eta'),u_{\eta,\eta'}} - \pi_{(0,\eta,\eta')} \circ D_{(0,\eta,\eta'),u_{\eta,\eta'}}) \circ Q'_{(\xi,\eta,\eta')} s \\ & + \left(\pi_{(0,\eta,\eta')} \circ D_{(0,\eta,\eta'),u_{\eta,\eta'}} \circ Q'_{(0,\eta,\eta')}\right) p_{(\xi,\eta,\eta)} s - p_{(\xi,\eta,\eta)} s \\ & + p_{(\xi,\eta,\eta)} s - s, \end{aligned}$$

where

$$\| \pi_{(\xi,\eta,\eta')} \circ D_{(\xi,\eta,\eta'),u_{\eta,\eta'}} - \pi_{(0,\eta,\eta')} \circ D_{(0,\eta,\eta'),u_{\eta,\eta'}} \| \leq C_1|\xi|,$$

$\| p_{(\xi,\eta,\eta)} - Id \| \leq C_2|\xi|$, and $\| Q'_{(\xi,\eta,\eta')} \| \leq C_3|\xi|$ for all $(\xi,\eta,\eta') \in B_{\delta_2} \times D_d \times D'_{d'}$.

We also have

$$\begin{aligned} & \| \left(\pi_{(0,\eta,\eta')} \circ D_{(0,\eta,\eta'),u_{\eta,\eta'}} \circ Q'_{(0,\eta,\eta')}\right) p_{(\xi,\eta,\eta)} s - p_{(\xi,\eta,\eta)} s \| \\ \leq \;& C_4 \left(|\log(|\eta| + |\eta'|)|^{\frac{1}{p}-1} \right) \| p_{(\xi,\eta,\eta)} s \|_{L^p} \\ \leq \;& C_4 \left(|\log(|\eta| + |\eta'|)|^{\frac{1}{p}-1} \right) \| s \|_{L^p} \end{aligned}$$

where C_4 is the constant C in Proposition 6.30. Therefore,

$$\begin{aligned} & \| \left(\pi_{(\xi,\eta,\eta')} \circ D_{(\xi,\eta,\eta'),u_{\eta,\eta'}} \circ Q'_{(\xi,\eta,\eta')}\right) s - s \|_{L^p} \\ & \qquad \leq (C_1C_3 + C_4 + C_2)(|\xi| + |\log(|\eta| + |\eta'|)|^{\frac{1}{p}-1}) \| s \|_{L^p} . \end{aligned}$$

□

Corollary 6.31. *There exist $\delta_2, d_1, \ldots, d_{l_0}, d'_1, \ldots, d'_{l_1} > 0$ such that for every $(\xi,\eta,\eta') \in B_{\delta_2} \times D_d \times D'_{d'}$ there is a right inverse $Q_{(\xi,\eta,\eta')}$ of $\pi_{(\xi,\eta,\eta')} \circ D_{(\xi,\eta,\eta'),u_{\eta,\eta'}}$ such that the operator norm $\| Q_{(\xi,\eta,\eta')} \| \leq C$ for some constant C.*

Proof. By Proposition 6.29, there exist $\delta_2, d_1, \ldots, d_{l_0}, d'_1, \ldots, d'_{l_1} > 0$ sufficiently small such that

$$\| \left(\pi_{(\xi,\eta,\eta')} \circ D_{(\xi,\eta,\eta'),u_{\eta,\eta'}} \circ Q'_{(\xi,\eta,\eta')}\right) s - s \|_{L^p} \leq \frac{1}{2} \| s \|_{L^p} .$$

for any $(\xi,\eta,\eta') \in B_{\delta_2} \times D_d \times D'_{d'}$

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Let $A_{\xi,\eta,\eta'} = \pi_{(\xi,\eta,\eta')} \circ D_{(\xi,\eta,\eta'),u_{\eta,\eta'}} \circ Q'_{(\xi,\eta,\eta')} - I$, where I is the identity map. Then $\| A_{\xi,\eta,\eta'} \| \leq \frac{1}{2}$, so $I + A_{\xi,\eta,\eta'}$ is invertible and $\| (I + A_{\xi,\eta,\eta'})^{-1} \| \leq 2$. Let $Q_{(\xi,\eta,\eta')} = Q'_{(\xi,\eta,\eta')} \circ (I + A_{\xi,\eta,\eta'})^{-1}$, then $Q_{(\xi,\eta,\eta')}$ has the desired properties. □

Let $\delta_2, d_1, \ldots, d_{l_0}, d'_1, \ldots, d'_{l_1}$ be chosen as in Corollary 6.31. Then $\pi_{(\xi,\eta,\eta')} \circ D_{(\xi,\eta,\eta'),u_{\eta,\eta'}}$ is surjective for $(\xi, \eta, \eta') \in B_{\delta_2} \times D_d \times D'_{d'}$. We will construct a linear isomorphism

$$i_{(\xi,\eta,\eta')} : \mathrm{Ker}(\pi \circ D_u) \longrightarrow \mathrm{Ker}(\pi_{(\xi,\eta,\eta')} \circ D_{(\xi,\eta,\eta'),u_{\eta,\eta'}}).$$

Given $w \in W^{1,p}(\Sigma, \partial\Sigma, u^*TX, (u|_{\partial\Sigma})^*TL)$, we cut it off near nodes to obtain

$$g'_{(0,\eta,\eta')}(w) \in W^{1,p}(\Sigma_{(0,\eta,\eta')}, \partial\Sigma_{(0,\eta,\eta')}, u^*_{\eta,\eta'}TX, (u_{\eta,\eta'}|_{\partial\Sigma})^*TL).$$

We first look at the local model of an interior node. Let $u : B_{\epsilon_1,0} \to X$ be a stable map. We have constructed smooth maps $u_t : B_{\epsilon_1,t} \to X$ for small $t \in \mathbb{C}$ such that

$$u_t\left(z, \frac{t}{z}\right) = \begin{cases} u\left(0, \frac{t}{z}\right)) & \text{if } \frac{r^2}{\epsilon_1} < |z| < \frac{r\sqrt{r}}{2}, \\ p & \text{if } r\sqrt{r} \leq |z| \leq \sqrt{r}, \\ u(z,0) & \text{if } 2\sqrt{r} < |z| < \epsilon_1. \end{cases}$$

where $p = u(0,0)$.

Let $r = \sqrt{|t|}$ as before, and set $s = (4r)^{\frac{1}{3}}$. For $w \in W^{1,p}(B_{\epsilon_1,0}, u^*TX, (u|_{\partial B_{\epsilon_1,0}})^*TL)$, define $w_t \in W^{1,p}(B_{\epsilon_1,t}, u^*TX, (u_t|_{\partial B_{\epsilon_1,t}})^*TL)$ by

$$w_t\left(z, \frac{t}{z}\right) = \begin{cases} (1 - \chi_s(\frac{t}{z}))w(0, \frac{t}{z}) + \chi_s(\frac{t}{z})P(z)w(0,0) & \text{if } \frac{r^2}{\epsilon_1} < |z| \leq r \\ (1 - \chi_s(z))w(z,0) + \chi_s(z)P(z)w(0,0) & \text{if } r \leq |z| < \epsilon_1 \end{cases}$$

where χ_s is the cutoff function in Lemma 6.25, and $P(z)$ is the parallel transport along the unique length minimizing geodesic from p to $u_t(z, \frac{t}{z})$. We have

$$w_t\left(z, \frac{t}{z}\right) = \begin{cases} w(0, \frac{t}{z}) & \text{if } \frac{r^2}{\epsilon_1} < |z| \leq 2^{-\frac{2}{3}} r^{\frac{5}{3}} \\ P(z)w(0,0) & \text{if } \frac{r\sqrt{r}}{2} \leq |z| \leq 2\sqrt{r} \\ w(z,0) & \text{if } (4r)^{\frac{1}{3}} \leq |z| < \epsilon_1 \end{cases}$$

We apply above construction to each interior node and similar construction to each boundary node to obtain a linear map

$$g'_{(0,\eta,\eta')} : W^{1,p}(\Sigma, \partial\Sigma, u^*TX, (u|_{\partial\Sigma})^*TL) \to W^{1,p}(\Sigma_{(0,\eta,\eta')}, \partial\Sigma_{(0,\eta,\eta')}, u^*_{\eta,\eta'}TX, (u_{\eta,\eta'}|_{\partial\Sigma_{(0,\eta,\eta')}})^*TL),$$

which can also be viewed as a map

$$g'_{(\xi,\eta,\eta')} : W^{1,p}(\Sigma, \partial\Sigma, u^*TX, (u|_{\partial\Sigma})^*TL) \to W^{1,p}(\Sigma_{(\xi,\eta,\eta')}, \partial\Sigma_{(\xi,\eta,\eta')}, u^*_{\eta,\eta'}TX, (u_{\eta,\eta'}|_{\partial\Sigma_{(\xi,\eta,\eta')}})^*TL).$$

The restriction of $g'_{(\xi,\eta,\eta')}$ to $\mathrm{Ker}(\pi \circ D_u)$ is injective by the unique continuity theorem ([2]). Lemma 6.26 implies the following estimate for $w \in \mathrm{Ker}(\pi \circ D_u)$:

$$\| \pi_{(\xi,\eta,\eta')} \circ D_{u_{\eta,\eta'}} \circ g'_{(\xi,\eta,\eta')} w \|_{W^{1,p}} \leq C(|\xi| + \log(|\eta| + |\eta'|)^{\frac{1}{p}-1}) \| w \|_{W^{1,p}} .$$

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Let

$$i_{(\xi,\eta,\eta')} = (Id - Q_{(\xi,\eta,\eta')} \circ \pi_{(\xi,\eta,\eta')} \circ D_{(\xi,\eta,\eta'),u_{\eta,\eta'}}) \circ g'_{(\xi,\eta,\eta')}$$

Then $i_{(\xi,\eta,\eta')} : \mathrm{Ker}(\pi \circ D_u) \to \mathrm{Ker}(\pi_{(\xi,\eta,\eta')} \circ D_{(\xi,\eta,\eta'),u_{\eta,\eta'}})$ is injective for (ξ,η,η') sufficiently small. It is actually a linear isomorphism since

$$\dim \mathrm{Ker}(\pi_{(\xi,\eta,\eta')} \circ D_{(\xi,\eta,\eta'),u_{\eta,\eta'}}) = \mathrm{Ind} D_u + \dim E_\rho = \dim \mathrm{Ker}(\pi \circ D_u).$$

We are now ready to find exact solutions near the approximate solution $u_{\eta,\eta'}$.

Proposition 6.32. *There exist $\delta_2, d_1, \ldots, d_{l_0}, d'_1, \ldots, d'_{l_1}, \epsilon_1, \epsilon_2 > 0$ sufficiently small such that for all $(\xi,\eta,\eta') \in B_{\delta_2} \times D_d \times D_{d'}$, if*

$$w \in \mathrm{Ker}(\pi_{(\xi,\eta,\eta')} \circ D_{(\xi,\eta,\eta'),u_{\eta,\eta'}}), \quad \| w \|_{W^{1,p}} \leq \epsilon_1,$$

then there exists a unique

$$h(\xi,\eta,\eta',w) \in L^p(\Sigma_{(\xi,\eta,\eta')}, \Lambda^{0,1}_{\Sigma_{(\xi,\eta,\eta')}} \otimes u^*_{\eta,\eta'} TX)$$

such that

$$\pi_{(\xi,\eta,\eta')} \circ \bar\partial_{J,\Sigma_{(\xi,\eta,\eta')}} \exp_{u_{\eta,\eta'}}(w + Q_{(\xi,\eta,\eta')} h(\xi,\eta,\eta',w)) = 0$$

and

$$\| h(\xi,\eta,\eta',w) \|_{L^p} \leq \epsilon_2.$$

Proof. We assume that $(\xi,\eta,\eta') \in B_{\delta_2} \times D_d \times D'_{d'}$, and $\| w \|_{W^{1,p}} \leq \epsilon_1$, where δ_2, $d = (d_1, \ldots, d_{l_0})$, $d' = (d'_1, \ldots, d'_{l_1})$, and ϵ_1 will be determined later.

We will use Newton's method to find $h(\xi,\eta,\eta',w)$ as in [31, Theorem 3.3.4]. For convenience, we write v for $u_{\eta,\eta'}$, Q for $Q_{(\xi,\eta,\eta')}$, and $\bar\partial$ for $\bar\partial_{J,\Sigma_{(\xi,\eta,\eta')}}$; we set $h_0 = 0$, and set

$$h_{n+1} = h_n - P_n \circ \pi \circ \bar\partial \exp_v(w + Qh_n)$$

where P_n is the parallel transport along the geodesic $t \in [0,1] \mapsto \exp_v((1-t)(w + Qh_n))$. Let D_n denote the linearization of $\bar\partial_{J,\Sigma_{(\xi,\eta,\eta')}}$ at $v_n = \exp_v(w + Qh_n)$, and write π_n for $\pi_{(\lambda_{(\xi,\eta,\eta')}, v_n)}$. We have

$$\begin{aligned}
&P_{n+1} \circ \pi \circ \bar\partial \exp_v(w + Qh_{n+1}) \\
&= P_{n+1} \circ \pi \circ \bar\partial \exp_v(w + Qh_n - Q \circ P_n \circ \pi \circ \bar\partial \exp_v(w + Qh_n)) \\
&= P_n \circ \pi \circ \bar\partial \exp_v(w + Qh_n) \\
&\quad - P_n \circ \pi_n \circ D_n \circ (d\exp_v)(w + Qh_n)(Q \circ P_n \circ \pi \circ \bar\partial \exp_v(w + Qh_n)) \\
&\quad + R(\pi \circ \bar\partial \exp_v(w + Qh_n)),
\end{aligned}$$

where

$$\| R(\pi \circ \bar\partial \exp_v((w + Qh_n) \|_{L^p} \leq C_1 \| \pi \circ \bar\partial \exp_v((w + Qh_n)) \|^2_{L^p},$$

$$\| P_n \circ \pi_n \circ D_n \circ (d\exp_v)(w + Qh_n) - \pi \circ D_{(\xi,\eta,\eta'),v} \| \leq C_2(|w| + |Qh_n|).$$

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Therefore,

$$\begin{aligned}&\| \pi \circ \bar\partial \exp_v(w + Qh_{n+1}) \|_{L^p} \\ &\qquad \le C_3 \left(\| w \|_{C^0} + \| Qh_n \|_{C^0} + \| \pi \circ \bar\partial \exp_v(w+Qh_n) \|_{L^p}\right) \| \pi \circ \bar\partial \exp_v(w+Qh_n) \|_{L^p} \;.\end{aligned}$$

We also have

$$h_n = -\sum_{k=0}^{n-1} P_k \circ \pi \circ \bar\partial \exp_v(w + Qh_k),$$

thus

$$\begin{aligned}&\| \pi \circ \bar\partial \exp_v(w + Qh_{n+1}) \|_{L^p} \\ &\qquad \le C_3 \left(\| w \|_{W^{1,p}} + \sum_{k=0}^{n} \| \pi \circ \bar\partial \exp_v(w+Qh_k) \|_{L^p}\right) \| \pi \bar\partial \exp_v(w+Qh_n) \|_{L^p} \;.\end{aligned}$$

Let $a_n = \| \pi \circ \bar\partial \exp_v(w+Qh_n) \|_{L^p} \ge 0$ and $b = \| w \|_{W^{1,p}} \ge 0$, and arrange that

$$a_{n+1} \le C_3 \left(b + \sum_{k=0}^{n} a_k\right) a_n$$

We will show that if $a_0, b \le \frac{1}{6C_3}$ then $a_{n+1} \le \frac{1}{2} a_n$. We prove this by induction. For $n = 0$, we have

$$a_1 \le C_3(b + a_0)a_0 \le C_3\left(\frac{1}{6C_3} + \frac{1}{6C_3}\right) a_0 = \frac{1}{3}a_0 \le \frac{1}{2}a_0.$$

Now suppose that $a_{n+1} \le a_n$ for $n = 0, 1, \ldots, m$. Then

$$\sum_{k=0}^{m+1} a_k \le 2a_0,$$

thus

$$a_{m+2} \le C_3(b + 2a_0)a_{m+1} \le C_3\left(\frac{1}{6C_3} + \frac{1}{3C_3}\right) a_{m+1} = \frac{1}{2}a_{m+1}.$$

Let $D = D_{(\xi,\eta,\eta'),u_{\eta,\eta'}}$. Then

$$\pi \circ \bar\partial \exp_v w = \pi \circ \bar\partial v + \pi \circ Dw + R(w),$$

where $\| R(w) \|_{L^p} \le C_4 \| w \|_{C^0} \| w \|_{W^{1,p}}$, and $\pi \circ Dw = 0$. The proof of Lemma 6.23 can be modified to show that

$$\| \pi \circ \bar\partial v \|_{L^p} \le C_5 \left(|\xi| + (|\eta| + |\eta'|)^{\frac{1}{2p}}\right).$$

Therefore,

$$\begin{aligned} a_0 &= \| \pi \circ \bar\partial \exp_v w \|_{L^p} \\ &\le C_5 \left(|\xi| + (|\eta| + |\eta'|)^{\frac{1}{2p}}\right) + C_6 \| w \|^2_{W^{1,p}} \\ &\le C_5 \left(\delta_2 + (|d| + |d'|)^{\frac{1}{2p}}\right) + C_6 \epsilon_1^2, \end{aligned}$$

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which is less than $\frac{1}{6C_3}$ for sufficiently small $\delta_2, d, d', \epsilon_1$. $b = \| w \|_{W^{1,p}} \leq \epsilon_1$, which is in turn less than $\frac{1}{6C_3}$ for small ϵ_1. Hence if $\delta_2, d, d', \epsilon_1$ are sufficiently small, h_n converges uniformly in L^p norm, and the limit $h(\xi, \eta, \eta', w)$ satisfies

$$\pi \circ \bar{\partial} \exp_v(w + Qh(\xi, \eta, \eta', w)) = 0,$$

$$\| h(\xi, \eta, \eta', w) \|_{L^p} \leq 2a_0 \leq 2C_5(\delta_2 + (|d| + |d'|)^{\frac{1}{2p}}) + 2C_6\epsilon_1^2.$$

This proves the existence of $h(\xi, \eta, \eta', w)$.

We now show the uniqueness of $h(\xi, \eta, \eta', w)$. Write s_1 for $h(\xi, \eta, \eta', w)$, and suppose that s_2 satisfies

$$\pi \circ \bar{\partial} \exp_v(w + Qs_2) = 0$$

and

$$\| s_2 \|_{L^p} \leq 2C_5(\delta_2 + (|d| + |d'|)^{\frac{1}{2p}}) + 2C_6\epsilon_1^2.$$

Let P_1, P_2 denote the parallel transports along the geodesics

$$t \in [0, 1] \mapsto \exp_v((1 - t)(w + Qs_1)), \exp_v((1 - t)(w + Qs_2)),$$

respectively. Let D' denote the linearization of $\bar{\partial}_{J, \Sigma_{(\xi,\eta,\eta')}}$ at $v' = \exp_v(w + Qs_1)$, and let π' denote $\pi_{(\lambda_{(\xi,\eta,\eta')}, v')}$. We have

$$\begin{aligned}
0 &= P_2 \circ \pi \circ \bar{\partial} \exp_v(w + Qs_2) \\
&= P_2 \circ \pi \circ \bar{\partial} \exp_v(w + Qs_1 + Q(s_2 - s_1)) \\
&= P_1 \circ \pi \circ \bar{\partial} \exp_v(w + Qs_1) + P_1 \circ \pi' \circ D' \circ (d \exp_v)(w + Qs_1) \circ Q(s_2 - s_1) \\
&\quad + R(s_2 - s_1) \\
&= P_1 \circ \pi' \circ D' \circ (d \exp_v)(w + Qs_1) \circ Q(s_2 - s_1) + R(s_2 - s_1),
\end{aligned}$$

$$s_1 - s_2 = \big(P_1 \circ \pi' \circ D' \circ (d \exp_v)(w + Qs_1) - \pi \circ D\big) \circ Q(s_2 - s_1) + R(s_2 - s_1),$$

where

$$\| P_1 \circ \pi' \circ D' \circ (d \exp_v) - \pi \circ D \| \leq C_2(|w| + |Qs_1|),$$

and

$$\| R(s_2 - s_1) \|_{L^p} \leq \| s_2 - s_1 \|_{L^p}^2 .$$

Therefore,

$$\begin{aligned}
\| s_1 - s_2 \|_{L^p} &\leq C_7(\| w \|_{W^{1,p}} + \| s_1 \|_{L^p} + \| s_2 \|_{L^p}) \| s_1 - s_2 \|_{L^p} \\
&\leq C_7 \left(\epsilon_1 + 4C_5 \left(\delta_2 + (|d| + |d'|)^{\frac{1}{2p}}\right) + 4C_6\epsilon_1^2\right) \| s_1 - s_2 \|_{L^p},
\end{aligned}$$

where

$$C_7 \left(\epsilon_1 + 4C_5 \left(\delta_2 + (|d| + |d'|)^{\frac{1}{2p}}\right) + 4C_6\epsilon_1^2\right) \leq \frac{1}{2}$$

for sufficiently small $\delta_2, d, d', \epsilon_1$. We conclude that $s_1 = s_2$.

The proposition holds if $\delta_2, d_1, \ldots, d_{l_0}, d'_1, \ldots, d'_{l_1}, \epsilon_1, \epsilon_2 > 0$ are chosen such that

$$C_5 \left(\delta_2 + (|d| + |d'|)^{\frac{1}{2p}}\right) + C_6\epsilon_1^2 \leq \frac{1}{6C_3}, \quad \epsilon_1 \leq \frac{1}{6C_3},$$

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$$C_7\left(\epsilon_1 + 4C_5\left(\delta_2 + (|d| + |d'|)^{\frac{1}{2p}}\right) + 4C_6\epsilon_1^2\right) \le \frac{1}{2},$$

and $\epsilon_2 = 2C_5(\delta_2 + (|d| + |d'|)^{\frac{1}{2p}}) + 2C_6\epsilon_1^2$. □

6.4.3. *Kuranishi neighborhood.* Let $\delta_2, d_1, \dots, d_{l_0}, d'_1, \dots, d'_{l_1} > 0$ be chosen as in Proposition 6.32. Let $V_{\rho,\mathrm{map}} \subset H_{\rho,\mathrm{map}} = \mathrm{Ker}(\pi \circ D_u)$ be a neighborhood of the origin such that

$$\| i_{(\xi,\eta,\eta')} w \|_{W^{1,p}} < \epsilon_1 \quad \text{for all } w \in V_{\rho,\mathrm{map}}, (\xi,\eta,\eta') \in B_{\delta_2} \times D_d \times D'_{d'}.$$

Write $\tilde{B}$ for $B_{\delta_2} \times D_d \times D'_{d'}$, as at the beginning of Section 6.4.2. Define a map

$$\begin{aligned} \psi : \tilde{B} \times V_{\rho,\mathrm{map}} &\longrightarrow W_B \\ (\xi,\eta,\eta',w) &\mapsto [(\lambda_{(\xi,\eta,\eta')}, u_{(\xi,\eta,\eta',w)})] \end{aligned}$$

where W_B is defined as at the beginning of Section 6.4.2, and

$$u_{(\xi,\eta,\eta',w)} = \exp_{u_{\eta,\eta'}}\left(i_{(\xi,\eta,\eta')} w + Q_{(\xi,\eta,\eta')} h(\xi,\eta,\eta', i_{(\xi,\eta,\eta')} w)\right).$$

Then

$$\pi_{(\lambda_{(\xi,\eta,\eta')}, u_{(\xi,\eta,\eta',w)})} \circ \bar{\partial}_{J,\Sigma_{(\xi,\eta,\eta')}} u_{(\xi,\eta,\eta',w)} = 0,$$

so $(\xi,\eta,\eta',w) \mapsto \bar{\partial}_{J,\Sigma_{(\xi,\eta,\eta')}} u_{(\xi,\eta,\eta',w)}$ defines a map $s : \tilde{B} \times V_{\rho,\mathrm{map}} \to E_\rho$ such that $\psi(s^{-1}(0)) \subset \overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$. Actually, $\psi(s^{-1}(0))$ contains a neighborhood of ρ in $\overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$. To see this, note that any $\rho' \in \overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$ which is sufficiently close to ρ in the C^∞ topology can be written in the form

$$\rho' = (\lambda_{(\xi,\eta,\eta')}, \exp_{u_{\eta,\eta'}}(w))$$

where $w \in W^{1,p}(\Sigma_{(\xi,\eta,\eta')}, \partial\Sigma_{(\xi,\eta,\eta')}, u^*_{\eta,\eta'}TX, (u_{\eta,\eta'}|_{\partial\Sigma_{(\xi,\eta,\eta')}})^*TL)$ is small. There exist unique $w_0 \in \mathrm{Ker}(\pi_{(\xi,\eta,\eta')} \circ D_{(\xi,\eta,\eta'),u_{\eta,\eta'}})$ and $h \in L^p(\Sigma_{(\xi,\eta,\eta')}, \Lambda^{0,1}\Sigma_{(\xi,\eta,\eta')} \otimes u^*_{\eta,\eta'}TX)$ such that $w = w_0 + Q_{(\xi,\eta,\eta')} h$. We may further assume that w is small enough such that $w_0 = i_{(\xi,\eta,\eta')} w_1$ for some $w_1 \in V_{\rho,\mathrm{map}}$, and $\| h \|_{L^p} \le \epsilon_2$. Since

$$\bar{\partial}_{J,\Sigma_{(\xi,\eta,\eta')}} \exp_{u_{\eta,\eta'}}(w_0 + Q_{(\xi,\eta,\eta')} h) = 0,$$

we have $h = h(\xi,\eta,\eta',w_0)$ by the uniqueness part of Proposition 6.32. Hence $\rho' = \psi(\xi,\eta,\eta',w_1)$, and $s(\xi,\eta,\eta',w_1) = 0$.

Now assume that λ is stable, so that $\mathrm{Aut}\,\lambda$ is finite, and $V'_{\rho,\mathrm{map}} = V_{\rho,\mathrm{map}}$. Then there is an isomorphism $\phi : \lambda_{(\xi_1,\eta_1,\eta'_1)} \to \lambda_{(\xi_2,\eta_2,\eta'_2)}$ if and only if $(\xi_2,\eta_2,\eta'_2) = (\phi' \cdot \xi_1, \phi' \cdot \eta_1, \phi' \cdot \eta'_1)$ for some $\phi' \in \mathrm{Aut}\,\lambda$, and the action of ϕ' on the universal family restricts to ϕ on $\lambda_{(\xi_1,\eta_1,\eta'_1)}$. We may choose $\tilde{B}$, $V_{\rho,\mathrm{map}}$ small enough such that if $u_{(\xi,\eta,\eta',w_1)} = u_{(\phi\cdot\xi,\phi\cdot\eta,\phi\cdot\eta',w_2)} \circ \phi$ for $(\xi,\eta,\eta') \in \tilde{B}$, $w_1, w_2 \in V_{\rho,\mathrm{map}}$, $\phi \in \mathrm{Aut}\,\lambda$, then $\phi \in \mathrm{Aut}\,\rho$.

Given $\phi \in \mathrm{Aut}\,\rho$ and $w \in V_{\rho,\mathrm{map}}$, let $\phi_{(\xi,\eta,\eta')} w$ be the unique vector in $V_{\rho,\mathrm{map}}$ such that $u_{(\xi,\eta,\eta',w)} \circ \phi^{-1} = u_{(\phi\cdot\xi,\phi\cdot\eta,\phi\cdot\eta',\phi_{(\xi,\eta,\eta')} w)}$. Then $\mathrm{Aut}\,\rho$ acts on $\tilde{B} \times V_{\rho,\mathrm{map}}$ by $\phi\cdot(\xi,\eta,\eta',w) = (\phi\cdot\xi, \phi\cdot\eta, \phi\cdot\eta', \phi_{(\xi,\eta,\eta')} w)$. From the above discussion, $\psi(\xi_1,\eta_1,\eta'_1,w_1) = \psi(\xi_2,\eta_2,\eta'_2,w_2)$ if and only if $(\xi_2,\eta_2,\eta'_2,w_2) = \phi \cdot (\xi_1,\eta_1,\eta'_1,w_1)$ for some $\phi \in \mathrm{Aut}\,\rho$. Let V_ρ be an $\mathrm{Aut}\,\rho$-invariant neighborhood of the origin in $\tilde{B} \times$

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$V_{\rho,\text{map}} \subset H_{\rho,\text{domain}} \times H_{\rho,\text{map}} = H_\rho$. Let $s_\rho : V_\rho \to E_\rho$ be the restriction of s. Then s_ρ is $\mathrm{Aut}\,\rho$-equivariant. The restriction of ψ gives a continuous map $\psi_\rho : s_\rho^{-1}(0) \to \overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$ such that $s_\rho^{-1}(0)/\mathrm{Aut}\,\rho \to \overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$ is injective. It is actually a homeomorphism onto a neighborhood of ρ in $\overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$ since the C^∞ and $W^{1,p}$ topologies are equivalent on $\overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$. This completes the proof of Theorem 6.19 when the domain λ of ρ is stable.

Remark 6.33. *The section s in the above construction is only continuous, not smooth. It is not hard to show that it is smooth within the stratum, but its dependence on η, η' is not even C^1 (see Lemma 6.23). This is because the smooth structure of V_ρ is canonical within the stratum but dependent on our particular gluing construction in the direction (η, η') transversal to the stratum.*

If $\lambda = (\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h)$ is not stable, we add minimal number of marked points to obtain a stable marked bordered Riemann surface

$$\tilde{\lambda} = (\Sigma, \mathbf{B}, \tilde{\mathbf{p}}; \tilde{\mathbf{q}}^1, \ldots, \tilde{\mathbf{q}}^h)$$

where $\tilde{\mathbf{p}} = (p_1, \ldots, p_{n+\hat{n}})$, $\tilde{\mathbf{q}}^i = (q_1^i, \ldots, q^i_{m^i+\hat{m}^i})$, $p_{n+1}, \ldots, p_{n+\hat{n}}$ are additional interior marked points, and $q^i_{m^i+1}, \ldots, q^i_{m^i+\hat{m}^i}$ are additional marked points on B^i. Note that when counting the above minimal number, an interior marked point counts twice, while a boundary marked point counts once. We have

$$2\hat{n} + \hat{m}^1 + \cdots \hat{m}^h = \dim_{\mathbb{R}} H_{\rho,\text{aut}}.$$

Let $\tilde{\rho} = (\tilde{\lambda}, u) \in \overline{M}_{(g,h),(n+\hat{n},\vec{m}+\vec{\hat{m}})}(X, L \mid \beta, \vec{\gamma}, \mu)$. Then $H_{\tilde{\rho},\text{domain}} = H_{\rho,\text{domain}}$ and $H_{\tilde{\rho},\text{aut}} = 0$. If the additional marked points are chosen in $K_{3\sqrt{|d|+|d'|}}(\Sigma)$, then the construction of $\lambda_{(\xi,\eta,\eta')}$ also yields deformation $\tilde{\lambda}_{(\xi,\eta,\eta')}$ of $\tilde{\lambda}$ for $(\xi, \eta, \eta') \in B_{\delta_2} \times D_d \times D'_{d'} = \tilde{B} \subset H_{\tilde{\rho},\text{domain}} = H_{\rho,\text{domain}}$. Both $\mathrm{Aut}\,\tilde{\lambda}$ and $\mathrm{Aut}\,\rho$ are subgroups of $\mathrm{Aut}\,\lambda$, and $\mathrm{Aut}\,\tilde{\rho} = \mathrm{Aut}\,\tilde{\lambda} \cap \mathrm{Aut}\,\rho$. We may choose $E_{\tilde{\rho}} = E_\rho$, so that $H_{\tilde{\rho},\text{map}} = H_{\rho,\text{map}}$, $H_{\tilde{\rho}} = H_\rho$.

There is a map

$$F : \overline{M}_{(g,h),(n+\hat{n},\vec{m}+\vec{\hat{m}})}(X, L \mid \beta, \vec{\gamma}, \mu) \to \overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$$

defined by forgetting $p_{n+1}, \ldots, p_{n+\hat{n}}, q_{m+1}, \ldots, q_{m+\hat{m}}$ and then contracting non-stable components which are mapped to points. We have $F(\tilde{\rho}) = \rho$. We will construct a multi-valued map A from a neighborhood U_ρ of ρ in $\overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$ to $\overline{M}_{(g,h),(n+\hat{n},\vec{m}+\vec{\hat{m}})}(X, L \mid \beta, \vec{\gamma}, \mu)$ such that $F \circ A$ is the identity map, and $A(\rho) = \tilde{\rho}$.

The additional marked points are on unstable components, where u is not a constant. Let $2N$ be the dimension of X as before. For $j = n+1, \ldots, n+\hat{n}$, we may assume that there is a geodesic ball in X centered at $u(p_j)$ such that its intersection with the image of u is an embedded holomorphic disc D_j, and $u^{-1}(D_j) \to D_j$ is a trivial cover. $u^{-1}(D_j)$ might consist of more than one connected components if u is not injective. Let $B_j \subset X$ be an embedded $(2N-2)$-dimensional ball which is the image of N_{p_j} under $\exp_{u(p_j)}$, where N_{p_j} is a small ball centered at the origin in the orthogonal complement of $u_*(T_{p_j}\Sigma)$ in $T_{u(p_j)}X$. Then D_j and B_j intersect orthogonally at $u(p_j)$. We choose N_{p_j} sufficiently small such that B_j intersects the image of u at a single point $u(p_j)$.

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Similarly, for $j^i = m^i+1,\dots,m^i+\hat{m}^i$, we may assume that there is a geodesic ball in X centered at $u(q_{j^i})$ such that its intersection with the image of u is an embedded holomorphic half disc $D^+_{j^i}$, and $u^{-1}(D^+_{j^i}) \to D^+_{j^i}$ is a trivial cover. Let $B'_{j^i} \subset L$ be an embedded $(N-1)$-dimensional ball which is the image of $N'_{q_{j^i}}$ under $\exp_{u(q_{j^i})}$, where $N'_{q_{j^i}}$ is a small ball centered at the origin in the orthogonal complement of $u_*(T_{q_{j^i}}\partial\Sigma)$ in $T_{u(q_{j^i})}L$. Then $I_{j^i} = D^+_{j^i} \cap L \cong [0,1]$ and B'_{j^i} intersect orthogonally in L at $u(q_{j^i})$. We choose $N'_{q_{j^i}}$ sufficiently small such that B'_{j^i} intersects the image of u at a single point $u(q_{j^i})$.

In a small neighborhood of ρ in $\overline{M}_{(g,h),(n,\vec{m})}(X,L \mid \beta,\vec{\gamma},\mu)$, intersecting with B_j for $j = n+1,\dots,n+\hat{n}$ and B'_{j^i} for $j^i = m^i+1,\dots,m^i+\hat{m}^i$ determines additional marked points and gives the desired multi-valued map $A : U_\rho \to \overline{M}_{(g,h),(n+\hat{n},\vec{m}+\vec{\hat{m}})}(X,L \mid \beta,\vec{\gamma},\mu)$ for some neighborhood of ρ in $\overline{M}_{(g,h),(n,\vec{m})}(X,L \mid \beta,\vec{\gamma},\mu)$. A is single-valued if $\mathrm{Aut}\,\tilde{\rho} = \mathrm{Aut}\,\rho$. Let

$$\begin{aligned}\tilde{U}'_\rho &= \Big\{\tilde{\rho}' = (\Sigma',\mathbf{B}';\mathbf{p}';(\mathbf{q}')^1,\dots,(\mathbf{q}')^h;u') \in F^{-1}(U_\rho) \mid \\ &\quad u'(p_j) \in B_j \text{ for } j = n+1,\dots,n+\hat{n},\ u'(q_{j^i}) \in B'_{j^i} \text{ for } j^i = m^i+1,\dots,m^i+\hat{m}^i\Big\},\end{aligned}$$

and let $\tilde{U}_\rho$ be the connected component containing $\tilde{\rho}$. Then $\tilde{U}_\rho = A(U_\rho)$, and the fiber of $\tilde{U}_\rho \to U_\rho$ is finite.

Let $\hat{W}$ be the (real) codimension $(2\hat{n}+\hat{m}^1+\cdots+\hat{m}^h)$ subspace of $W^{1,p}(\Sigma,\partial\Sigma,u^*TX,(u|_{\partial\Sigma})^*TL)$ defined by

$$\hat{W} = \left\{ w \in W^{1,p}(\Sigma,\partial\Sigma,u^*TX,(u|_{\partial\Sigma})^*TL) \;\middle|\; \begin{array}{ll} w(p_j) \in T_{u(p_j)}B_j & \text{for } j = n+1,\dots,n+\hat{n} \\ w(q_{j^i}) \in T_{u(q_{j^i})}B'_{j^i} & \text{for } j^i = m^i+1,\dots,m^i+\hat{m}^i \end{array} \right\}$$

Then $H_{\rho,\mathrm{aut}} \cap \hat{W} = \{0\}$, and

$$W^{1,p}(\Sigma,\partial\Sigma,u^*TX,(u|_{\partial\Sigma})^*TL) = H_{\rho,\mathrm{aut}} \oplus \hat{W}.$$

Let

$$\begin{aligned}\hat{W}_{(\xi,\eta,\eta')} &= \Big\{ w \in W^{1,p}(\Sigma_{(\xi,\eta,\eta')},\partial\Sigma_{(\xi,\eta,\eta')},u^*_{\eta,\eta'}TX,(u_{\eta,\eta'}|_{\partial\Sigma_{(\xi,\eta,\eta')}})^*TL)\Big| \\ &\quad w(p_j) \in T_{u(p_j)}B_j \text{ for } j = n+1,\dots,n+\hat{n} \\ &\quad w(q_{j^i}) \in T_{u(q_{j^i})}B'_{j'} \text{ for } j^i = m^i+1,\dots,m^i+\hat{m}^i\Big\}.\end{aligned}$$

Then

$$W^{1,p}(\Sigma_{(\xi,\eta,\eta')},\partial\Sigma_{(\xi,\eta,\eta')},u^*_{\eta,\eta'}TX,(u_{\eta,\eta'}|_{\partial\Sigma_{(\xi,\eta,\eta')}})^*TL) = i_{(\xi,\eta,\eta')}H_{\rho,\mathrm{aut}} \oplus \hat{W}_{(\xi,\eta,\eta')}$$

for (ξ,η,η') sufficiently small. Let

$$p_{(\xi,\eta,\eta')} : W^{1,p}(\Sigma_{(\xi,\eta,\eta')},\partial\Sigma_{(\xi,\eta,\eta')},u^*_{\eta,\eta'}TX,(u_{\eta,\eta'}|_{\partial\Sigma_{(\xi,\eta,\eta')}})^*TL) \to \hat{W}_{(\xi,\eta,\eta')}$$

be the projection. We have $\pi_{(\xi,\eta,\eta')}\circ D_{(\xi,\eta,\eta'),u_{\eta,\eta'}}\circ p_{(\xi,\eta,\eta')} = \pi_{(\xi,\eta,\eta')}\circ D_{(\xi,\eta,\eta'),u_{\eta,\eta'}}$, since $i_{(\xi,\eta,\eta')}H_{\rho,\mathrm{aut}} \subset \mathrm{Ker}(\pi_{(\xi,\eta,\eta')}\circ D_{(\xi,\eta,\eta'),u_{\eta,\eta'}})$. By replacing $Q_{(\xi,\eta,\eta')}$ with $p_{(\xi,\eta,\eta')}\circ Q_{(\xi,\eta,\eta')}$, we may assume that $\mathrm{Im}Q_{(\xi,\eta,\eta')} \subset \hat{W}_{(\xi,\eta,\eta')}$.

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Let $\hat{H}_{\tilde{\rho},\mathrm{map}} = H_{\tilde{\rho},\mathrm{map}} \cap \hat{W} \cong H'_{\rho,\mathrm{map}}$. We may modify the isomorphism $i_{(\xi,\eta,\eta')} : H_{\rho,\mathrm{map}} \to \mathrm{Ker}(\pi_{(\xi,\eta,\eta')} \circ D_{(\xi,\eta,\eta'),u_{\eta,\eta'}})$ such that

$$i_{(\xi,\eta,\eta')}\hat{H}_{\tilde{\rho},\mathrm{map}} = \mathrm{Ker}(\pi_{(\xi,\eta,\eta')} \circ D_{(\xi,\eta,\eta'),u_{\eta,\eta'}}) \cap \hat{W}_{(\xi,\eta,\eta')}.$$

For $(\xi,\eta,\eta',w) \in \tilde{B} \times V_{\tilde{\rho},\mathrm{map}} = \tilde{B} \times V_{\rho,\mathrm{map}}$, define $\tilde{\psi}(\xi,\eta,\eta',w) = (\tilde{\lambda}_{(\xi,\eta,\eta')}, u_{(\xi,\eta,\eta',w)})$. For $w \in \hat{W}$, we have $i_{(\xi,\eta,\eta')}w + Q_{(\xi,\eta,\eta')}h(\xi,\eta,\eta',i_{(\xi,\eta,\eta')}w) \in \hat{W}_{(\xi,\eta,\eta')}$, thus

$$u_{(\xi,\eta,\eta',w)} = \exp_{u_{\eta,\eta'}}(i_{(\xi,\eta,\eta')}w + Q_{(\xi,\eta,\eta')}h(\xi,\eta,\eta',i_{(\xi,\eta,\eta')}) \in \hat{W}_{(\xi,\eta,\eta')}$$

satisfies $u_{(\xi,\eta,\eta',w)}(p_j) \in B_j$ for $j = n+1,\ldots,n+\hat{n}$, and $u_{(\xi,\eta,\eta',w)}(q_{j'}) \in B'_{j'}$ for $j = n+1,\ldots,n+\hat{n}$. Therefore, $\tilde{\psi}(\xi,\eta,\eta',w) \in \tilde{U}_\rho$ if $w \in \hat{W}$, $s(\xi,\eta,\eta',w) = 0$, and ξ,η,η',w are sufficiently small.

Conversely, if $(\tilde{\lambda}',u')$ is a stable map near $\tilde{\rho}$ in $\tilde{U}_\rho$, then $\tilde{\lambda}' = \lambda_{(\xi,\eta,\eta')}$ for some $(\xi,\eta,\eta') \in \tilde{B}$, and $u' = \exp_{u_{\eta,\eta'}}(w)$ for some $w \in \hat{W}_{(\xi,\eta,\eta')}$. There exist unique $w_0 \in \mathrm{Ker}(\pi_{\xi,\eta,\eta'} \circ D_{(\xi,\eta,\eta'),u_{\eta,\eta'}}) \cap \hat{W}_{(\xi,\eta,\eta')}$ and $h \in L^p(\Sigma_{(\xi,\eta,\eta')}, \Lambda^{0,1}\Sigma_{(\xi,0,0)} \otimes u^*_{\eta,\eta'}TX)$ such that $w = w_0 + Q_{(\xi,\eta,\eta')}h$. We have $h = h(\xi,\eta,\eta',w_0)$ since

$$\bar{\partial}_{J,\Sigma_{(\xi,\eta,\eta')}} \exp_{u_{\eta,\eta'}}(w_0 + Q_{(\xi,\eta,\eta')}h) = 0.$$

Let $w_1 = i^{-1}_{(\xi,\eta,\eta')}w_0 \in \hat{H}_{\tilde{\rho},\mathrm{map}}$. Then $u' = u(\xi,\eta,\eta',w_1)$, and $s(\xi,\eta,\eta',w_1) = 0$. Let $\hat{V}_{\rho,\mathrm{map}} = V_{\rho,\mathrm{map}} \cap \hat{H}_{\tilde{\rho},\mathrm{map}}$. Then $\mathrm{Aut}\,\tilde{\rho}$ acts on $\tilde{B} \times \hat{V}_{\tilde{\rho},\mathrm{map}}$. Let $\hat{V}_{\tilde{\rho}}$ be an $\mathrm{Aut}\,\tilde{\rho}$ invariant neighborhood of the origin in $\tilde{B} \times \hat{V}_{\tilde{\rho},\mathrm{map}}$. It corresponds to a neighborhood V'_ρ of the origin in $\tilde{B} \times H'_{\rho,\mathrm{map}}$ under the isomorphism $\hat{H}_{\tilde{\rho},\mathrm{map}} \cong H'_{\rho,\mathrm{map}}$. We have the following commutative diagram

$$\begin{array}{ccc}
(s^{-1}(0) \cap \hat{V}_{\tilde{\rho}})/\mathrm{Aut}\,\tilde{\rho} & \xrightarrow{\hat{\psi}} & U_{\tilde{\rho}} \\
\downarrow & & \downarrow F \\
(s^{-1}(0) \cap V'_\rho)/\mathrm{Aut}\,\rho & \xrightarrow{\psi'} & U_\rho
\end{array}$$

where $\hat{\psi}$ and ψ' are injective. ψ' is a homeomorphism onto its image.

6.5. **Transition functions.** For each $\rho \in \mathcal{M} = \overline{M}_{(g,h),(n,\vec{m})}(X,L \mid \beta,\vec{\gamma},\mu)$ and each choice of E_ρ, we have constructed a Kuranishi neighborhood $(V'_\rho, E_\rho, \mathrm{Aut}\,\rho, \psi_\rho, s_\rho)$.

Remark 6.34. *A different choice of E_ρ yields a different, but equivalent Kuranishi neighborhood in the sense of Definition 6.2. Actually, if $E_{1,\rho}, E_{2,\rho} \subset L^p(\Sigma, \Lambda^{0,1} \otimes u^*TX)$ are two different choices, set $E_\rho = E_{1,\rho} + E_{2,\rho}$. Then $(V'_\rho, E_\rho, \mathrm{Aut}\,\rho, \psi_\rho, s_\rho)$ can serve as the $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ in Definition 6.2.*

In this section, we will modify the Kuranishi neighborhoods we constructed so that we can construct transition functions between them. We will follow [9, Section 15] closely to which we refer the reader for further details.

$\mathcal{M}$ is compact in C^∞ topology, so there exist $\rho_1,\ldots,\rho_l \in \mathcal{M}$ such that

$$\{U'_i = \psi_{\rho_i}(s^{-1}_{\rho_i}(0)) \mid i = 1,\ldots,l\}$$

is an open cover of $\mathcal{M}$, and there is $U_i \subset\subset U'_i$ such that $\{U_i \mid i = 1,\ldots,l\}$ is still an open cover of $\mathcal{M}$. $\mathcal{M}$ is compact and Hausdorff, so the closure K_i of U_i is compact for $i = 1,\ldots,l$.

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Let $E_i \to V_i = V_{\rho_i}$ be the obstruction bundle constructed from E_{ρ_i}. We may choose ρ_i and E_{ρ_i} such that if

$$\rho = (\Sigma, \mathbf{B}; \mathbf{p}; \mathbf{q}^1, \ldots, \mathbf{q}^h; u) \in K_{i_1} \cap \ldots \cap K_{i_k}$$

and $\rho \notin K_i$ if $i \neq i_k$, then the subspace $\tilde{E}_\rho$ of $L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes u^*TX)$ spanned by $(E_{i_1})_\rho, \ldots, (E_{i_k})_\rho$ is actually a direct sum $\tilde{E}_\rho = (E_{i_1})_\rho \oplus \cdots \oplus (E_{i_k})_\rho$. We use $\tilde{E}_\rho$ to construct a Kuranishi neighborhood $(\tilde{V}_\rho, \tilde{E}_\rho, \mathrm{Aut}\,\rho, \tilde{\psi}_\rho, \tilde{s}_\rho)$ as in Sections 6.4.2, 6.4.3. We shrink $\tilde{V}_\rho$ such that $\tilde{\psi}_\rho(\tilde{s}_\rho^{-1}(0)) \subset U_{i_1} \cap \cdots \cap U_{i_k} \cap K^c_{j_1} \cap \cdots \cap K^c_{j_{l-k}}$, where $\{i_1, \ldots, i_k, j_1, \ldots, j_{l-k}\} = \{1, \ldots, l\}$.

Suppose that $\rho' = \tilde{\psi}_\rho(\xi, \eta, \eta', w)$, where $(\xi, \eta, \eta', w) \in \tilde{V}_\rho$, $\tilde{s}_\rho(\xi, \eta, \eta', w) = 0$. Then $\mathrm{Aut}\,\rho'$ can be identified with the stabilizer $(\mathrm{Aut}\,\rho)_{(\xi,\eta.\eta',w)}$ of the action $\mathrm{Aut}\,\rho$ on $\tilde{V}_\rho$. This gives a monomorphism $h_{\rho\rho'} : \mathrm{Aut}\,\rho' \to \mathrm{Aut}\,\rho$. Without loss of generality, we may assume that $\rho' \in K_1 \cap \cdots \cap K_k$, and $\rho' \notin K_i$ for $i = k+1, \ldots, l$. We have $\rho \in K_1 \cap \cdots \cap K_k$. We may assume that $\rho \in K_1 \cap \cdots \cap K_{k'}$, and $\rho \notin K_i$ for $i = k'+1, \ldots, l$, where $k' \geq k$. It follows from the construction that there is an $\mathrm{Aut}\,\rho'$-invariant neighborhood $V_{\rho\rho'}$ of $0 = \psi_{\rho'}^{-1}(\rho')$ in $\tilde{V}_{\rho'}$ such that we have the following commutative diagram:

$$\begin{array}{ccc} \tilde{E}_{\rho'}|_{V_{\rho\rho'}} & \xrightarrow{\hat{\phi}_{\rho\rho'}} & \tilde{E}_\rho \\ \Big\downarrow & & \Big\downarrow \\ V_{\rho\rho'} & \xrightarrow{\phi_{\rho\rho'}} & \tilde{V}_\rho \end{array}$$

where $\hat{\phi}_{\rho\rho'}$ is induced by the inclusion $E_1 \oplus \cdots \oplus E_k \to E_1 \oplus \cdots \oplus E_{k'}$. Both $\hat{\phi}_{\rho\rho'}$ and $\phi_{\rho\rho'}$ are $h_{\rho\rho'}$-equivariant embedding of codimension $\mathrm{rank}E_{k+1} + \cdots + \mathrm{rank}E_{k'}$.

The Kuranishi neighborhoods $(\tilde{V}_\rho, \tilde{E}_\rho, \mathrm{Aut}\,\rho, \tilde{\psi}_\rho, \tilde{s}_\rho)$ satisfy the properties listed in Definition 6.1, and the transition functions $(V_{\rho\rho'}, \hat{\phi}_{\rho\rho'}, \phi_{\rho\rho'}, h_{\rho\rho'})$ satisfy the properties listed in Definition 6.3. From now on, we will write $(V'_\rho, E_\rho, \mathrm{Aut}\,\rho, \psi_\rho, s_\rho)$ for the $(\tilde{V}_\rho, \tilde{E}_\rho, \mathrm{Aut}\,\rho, \tilde{\psi}_\rho, \tilde{s}_\rho)$ we just constructed.

Remark 6.35. *The $\hat{\phi}_{\rho\rho'}, \phi_{\rho\rho'}$ in the above construction are smooth when restricted to a stratum. It is possible to refine the above modification of Kuranishi neighborhoods such that $\hat{\phi}_{\rho\rho'}, \phi_{\rho\rho'}$ are smooth (see* [9, Section 15]*). Such a refinement is artificial since our smooth structure in directions transversal to a stratum is not natural, as discussed in Remark 6.33. For simplicity of exposition, we will assume the smoothness of $\hat{\phi}_{\rho\rho'}, \phi_{\rho\rho'}$ in Section 7, though such an assumption is not absolutely necessary for our purposes.*

By Remark 6.34, the equivalence class of the Kuranishi structure constructed above is independent of various choices in our construction.

6.6. **Orientation.** Recall that the orientation bundle of the Kuranishi structure is the real line orbibundle obtained by gluing $\det(TV'_\rho) \times \det(E_\rho)^{-1}$. In our case,

$$(\det(TV'_\rho) \otimes \det(E_\rho)^{-1})_{(\xi,\eta,\eta',w)} \cong \det(\mathrm{Ind}D_{(\xi,\eta,\eta'),u_{\eta,\eta'}}) \otimes \det\left(\mathrm{Ind}(T_{\Sigma_{(\xi,\eta,\eta')}}, T_{\partial\Sigma_{(\xi,\eta,\eta')}})\right)^{-1}$$

where $\mathrm{Ind}(D)$ denotes the virtual real vector space $\mathrm{Ker}(D) - \mathrm{Coker}(D)$, as in Section 6.2.4.

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Theorem 6.36. *The Kuranishi structure constructed above is orientable if L is spin or if $h=1$ and L is relatively spin, i.e., L is orientable and $w_2(TL)=\alpha|_L$ for some $\alpha\in H^2(X,\mathbb{Z}_2)$.*

To orient the index of the linearization $D\bar{\partial}_J$ of $\bar{\partial}_J$ at a stable map, we need the following generalization of [10, Proposition 21.3].

Lemma 6.37. *Let $(E,E_{\mathbb{R}})\to(\Sigma,\partial\Sigma)$ be a Riemann-Hilbert bundle over a prestable bordered Riemann surface without boundary nodes Σ, and let $E_{\mathbb{R}}$ be a totally real subbundle of $E|_{\partial\Sigma}$. Then an ordering of the connected components of $\partial\Sigma$ and a trivialization of $E_{\mathbb{R}}$ determine an orientation of* $\mathrm{Ind}(E,E_{\mathbb{R}})$, *where* $\mathrm{Ind}(E,E_{\mathbb{R}})$ *is defined as in* [24, Definition 3.4.1].

Proof. Let $B^1,...,B^h$ be the ordered connected components of $\partial\Sigma$. An isomorphism $E_{\mathbb{R}}\cong\partial\Sigma\times\mathbb{R}^n$ is a collection of isomorphisms $E_{\mathbb{R}}|_{B^i}\cong B^i\times\mathbb{R}^n$. Let $\epsilon>0$ be such that $A_i=B(B^i,\epsilon)$, the collar neighborhood of B^i in Σ of radius ϵ w.r.t. some admissible metric on Σ, are disjoint. By tensoring with $\mathbb{C}$, we have trivializations $E|_{B^i}\cong B^i\times\mathbb{C}^n$. By deforming the Hermitian connection, we may assume that the connection is flat on $A=\bigcup_{i=1}^h B(B^i,\frac{2}{3}\epsilon)$ and there are parallel sections $s_1,...,s_h$ on $E|_A$ such that for $x\in\partial\Sigma$, $s_i(x)$ corresponds to (x,e_i) under the isomorphism $E_{\mathbb{R}}\cong\partial\Sigma\times\mathbb{R}^n$, where $e_1,\dots e_n$ are the standard basis of $\mathbb{R}^n$.

The boundary of $A_i=B(B^i,\frac{1}{2})$ is the disjoint union of two circles, B^i and $(B')^i$. We shrink $(B')^i$ to obtain a family of prestable bordered Riemann surfaces Σ_t, $t\in[0,1]$, such that $\Sigma_1=\Sigma$, Σ_t are homeomorphic to Σ, and Σ_0 is obtained from Σ by shrinking each $(B')^i$ to a point. $\Sigma_0=C\cup D^1\cup\ldots\cup D^h$ is a prestable bordered Riemann surface, where C is a complex algebraic curve of genus g, and D^i is a disc which intersects C at an interior node on Σ_0, for $i=1,\ldots,h$.

We extend $(E,E_{\mathbb{R}})$ to a family of Riemann-Hilbert bundles $(E(t),E_{\mathbb{R}}(t))\to(\Sigma_t,\partial\Sigma_t)$ such that $s_1,\ldots,s_h$ extend to a neighborhood of $\cup_{t\in[0,1]}\partial\Sigma_t$ in $\cup_{t\in[0,1]}\Sigma_t$ and give a holomorphic trivialization of $E(t)$ in a collar neighborhood of $\partial\Sigma_t$ and a trivialization of $E_{\mathbb{R}}(t)$. In particular, they are defined on $D^i\subset\Sigma_0$ to give an identification $(E(0),E_{\mathbb{R}}(0))|_{D_i}=(\mathbb{C}^n,\mathbb{R}^n)$.

We use the notation in [24, Section 3.4]. $\mathrm{Ind}(E(t),E_{\mathbb{R}}(t))$ is a family of virtual real vector spaces over $[0,1]$. We have

$$\mathrm{Ind}(E,E_{\mathbb{R}})=\mathrm{Ind}(E(1),E_{\mathbb{R}}(1))\cong\mathrm{Ind}(E(0),E_{\mathbb{R}}(0)),$$

so it suffices to orient $\mathrm{Ind}(E(0),E_{\mathbb{R}}(0))$. We have a long exact sequence

$$\begin{aligned}0\ &\to\ H^0(\Sigma_0,\partial\Sigma_0,E(0),E(0)_{\mathbb{R}})\to H^0(C,F)\oplus\bigoplus_{i=1}^h H^0(D^i,\partial D^i,\mathbb{C}^n,\mathbb{R}^n)\xrightarrow{e}\mathbb{C}^n\\ &\to\ H^1(\Sigma_0,\partial\Sigma_0,E(0),E(0)_{\mathbb{R}})\to H^1(C,F)\oplus\bigoplus_{i=1}^h H^1(D^i,\partial D^i,\mathbb{C}^n,\mathbb{R}^n)\to 0\end{aligned}$$

where $F=E(0)|_C$ is a holomorphic vector bundle of degree $\frac{1}{2}\mu(E,E_{\mathbb{R}})$, and e is given by

$$\begin{aligned}H^0(C,F)\oplus\bigoplus_{i=1}^h H^0(D^i,\partial D^i,\mathbb{C}^n,\mathbb{R}^n)\ &\to\ \mathbb{C}^n\\ (\xi_0,\xi_1,...,\xi_h)\ &\mapsto\ (\xi_0(p_1)-\xi_1(0),...,\xi_0(p_h)-\xi_h(0))\end{aligned}$$

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$p_i \in C$ and $0 \in D_i$ are identified to form an interior node of Σ_0. $E_{\mathbb{R}} \simeq \mathbb{R}^n$ gives the orientation on $H^0(D^i, \partial D^i, \mathbb{C}^n, \mathbb{R}^n) \simeq \mathbb{R}^n$ via the evaluation map. $H^1(D^i, \partial D^i, \mathbb{C}^n, \mathbb{R}^n) = 0$. $H^0(C, F), H^1(C, F)$ and $\mathbb{C}^n$ are complex vector spaces, thus canonically oriented. Therefore,

$$\mathrm{Ind}(E(0), E_{\mathbb{R}}(0)) = H^0(\Sigma_0, \partial\Sigma_0, E(0), E(0)_{\mathbb{R}}) - H^1(\Sigma_0, \partial\Sigma_0, E(0), E(0)_{\mathbb{R}})$$

is oriented. This orientation depends on the trivialization of $E_{\mathbb{R}}$ and the ordering of connected components of $\partial\Sigma$, since the trivialization of $E_{\mathbb{R}}$ determines the orientation on each $H^0(D^i, \partial D^i, \mathbb{C}^n, \mathbb{R}^n) \cong \mathbb{R}^n$ and the ordering of connected components of $\partial\Sigma$ determines the ordering of these h copies of $\mathbb{R}^n$. □

Proof of Theorem 6.36. It suffices to show that the orientation bundle is trivial when restricted to each loop $\gamma : S^1 \to \overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$, $\gamma(t) = \rho_t$. From the construction of Kuranishi structure we see that we may deform γ to a family of smooth maps $\tilde{\rho}_t \in V_{\rho_t}$ such that the domains of $\tilde{\rho}_t$ are smooth bordered Riemann surfaces. We first assume that these domains are stable. The tangent bundle of $\overline{M}_{(g,h),(n,m)}$ is orientable by Theorem 4.14, so it suffices to show that the index bundle $\mathrm{Ind}\, D\bar{\partial}$ is orientable along the loop $\tilde{\gamma}(t) = \tilde{\rho}_t$. Note that $\mathrm{Aut}\,\rho$ preserves the orientation of $\mathrm{Ind}\, D\bar{\partial}$ since it does not permute boundary components of the domain.

Let $\Phi : (\Sigma, \partial\Sigma) \times S^1 \to (X, L)$ be given by $\Phi(z, t) = u_t(z)$, where u_t is the map for ρ_t. Since the connected components of $\partial\Sigma$ are ordered, by Lemma 6.37 it suffices to show that $(\Phi|_{\partial\Sigma\times S^1})^* TL$ is stably trivial.

We first assume that L is relatively spin, i.e, L is orientable and $w_2(TL) = \alpha|_L$ for some $\alpha \in H^2(X, \mathbb{Z}_2)$. Choose a cellular decomposition on X such that $X^{(2)} \cap L = L^{(2)}$. There exists a real orientable vector bundle of rank 2 over $X^{(3)}$ such that $w_2(V) = \alpha|_{X^{(3)}} \in H^2(X^{(3)}, \mathbb{Z}_2)$. Then

$$w_2((TL \oplus V)|_{L^{(2)}}) = 0 \in H^2(L^{(2)}, \mathbb{Z}_2),$$

thus $(TL \oplus V)|_{L^{(2)}}$ is spin, and so stably trivializable on $L^{(2)}$.

We write

$$(TL \oplus V)|_{L^{(2)}} \oplus \mathbb{R}^k = \mathbb{R}^{N+k+2}.$$

Let $\tilde{\Phi}$ be homotopic to Φ such that $\tilde{\Phi}(\partial\Sigma \times S^1) \subset L^{(2)}$. It suffices to show that $(\tilde{\Phi}|_{\partial\Sigma\times S^1})^* TL$ is stably trivial. We have

$$(\tilde{\Phi}|_{\partial\Sigma\times S^1})^* TL \oplus (\tilde{\Phi}|_{\partial\Sigma\times S^1})^* V \oplus \mathbb{R}^k = \mathbb{R}^{N+k+2}$$

$$\begin{array}{ccc}
(\tilde{\Phi}|_{\partial\Sigma\times S^1})^* V & \longrightarrow & \tilde{\Phi}^* V \\
\downarrow & & \downarrow \\
\coprod_{i=1}^h R_i \times S^1 = \partial\Sigma \times S^1 & \xrightarrow{i} & \Sigma \times S^1
\end{array}$$

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We may view V as a complex line bundle. We need to show that $(\tilde{\Phi}|_{\partial\Sigma\times S^1})^*V$ is trivial, or equivalently, $n_i = (\deg \tilde{\Phi}|_{R_i\times S^1})^*V = 0$ for $i = 1, \dots, h$. We have

$$\begin{aligned}\sum_{i=1}^{h} n_i &= \deg(\tilde{\Phi}|_{\partial\Sigma\times S^1})^*V \\ &= (i_*[\partial\Sigma\times S^1])\cap c_1(\tilde{\Phi}^*V) \\ &= \partial(i_*[\Sigma\times S^1])\cap c_1(\tilde{\Phi}^*V) \\ &= 0.\end{aligned}$$

Thus $(\tilde{\Phi}|_{\partial\Sigma\times S^1})^*V$ is trivial if $h = 1$. For $h > 1$, we assume that L is spin, so that we may take $V = \mathbb{R}^2$, the trivial bundle.

We finally consider unstable cases:

(1) $(g, h) = (0, 1)$, $n = 1$, $\vec{m} = (0)$.
(2) $(g, h) = (0, 1)$, $n = 0$, $\vec{m} = (1)$.
(3) $(g, h) = (0, 1)$, $n = 0$, $\vec{m} = (2)$.
(4) $(g, h) = (0, 2)$, $n = 0$, $\vec{m} = (0, 0)$.

Cases (2) and (3) are treated in [10]. For (1) the domain is isomorphic to the unit disc with one interior marked point at the origin, and the automorphism group $U(1) = \{e^{i\theta} \mid \theta\in\mathbb{R}\}$ of the domain can be oriented by $\frac{\partial}{\partial\theta}$. For case (4), the domain is isomorphic to an annulus $\{z\in\mathbb{C} \mid 1\le |z|\le r\}$ for some $r\in(1,\infty)$, which is oriented by $\frac{\partial}{\partial r}$, and automorphism group $U(1)$ of the domain is oriented as above. □

7. Virtual fundamental chain

7.1. Construction of virtual fundamental chain.

We follow the general setting in Section 6.1.

Definition 7.1. *Let M be a Hausdorff space with a Kuranishi structure (with corners)*

$$\mathcal{K} = \left\{(V_p, E_p, \Gamma_p, \psi_p, s_p) : p\in M, (V_{pq}, \hat{\phi}_{pq}, \phi_{pq}, h_{pq}) : q \in \psi_p(s_p^{-1}(0))\right\}.$$

A Hausdorff topological space W is an ambient space *of $\mathcal{K}$ if*

(1) *M is a subspace of W.*
(2) *$\psi_p : V_p \to W$, $\psi_p(x)\in M$ if and only if $s_p(x) = 0$.*
(3) *$\psi_q = \psi_p\circ\phi_{pq}$.*
(4) *There is a subset $\partial W\subset W$ such that $\psi_p(x)\in\partial W$ if and only if $x\in\partial V_p$, where ∂V_p is the union of corners of V_p. We take $\partial W = \emptyset$ if $\mathcal{K}$ is a Kuranishi structure. We define $\partial M = M\cap\partial W$.*
(5) *$V_p/\Gamma_p \to W$ is injective.*

Remark 7.2. *If $\mathcal{K}_1$ and $\mathcal{K}_2$ are equivalent Kuranishi structures (with corners) in the sense of Definition 6.5, and they have the same ambient space W, we will implicitly assume that W is also an ambient space for the $\mathcal{K}$ in Definition 6.5, and $\psi_{i,p} = \psi_p\circ\phi_i : V_{i,p}\to W$ in Definition 6.2.*

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Example 7.3. *$\mathcal{W} = W^{1,p}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$ is an ambient space of the Kuranishi structure with corners on $\mathcal{M} = \overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$ constructed in Section 6. $\partial\mathcal{W} \subset \mathcal{W}$ is the subset corresponding to maps whose domain has at least one boundary node.*

Now assume that M is a *compact*, Hausdorff topological space with an *oriented* Kuranishi structure with corners $\mathcal{K}$, and that $\mathcal{K}$ has an ambient space W. Let d be the virtual dimension of $\mathcal{K}$. The *virtual fundamental chain* we shall construct is a singular d-chain $M_{\mathcal{K},\nu} \in \mathcal{S}_d(W, \mathbb{Q})$ such that $\partial M_{\mathcal{K},\nu} \in \mathcal{S}_{d-1}(\partial W, \mathbb{Q}) \subset \mathcal{S}_{d-1}(W, \mathbb{Q})$, so it represents a relative singular d-cycle $\bar{M}_{\mathcal{K},\nu} \in \mathcal{S}_d(W, \partial W, \mathbb{Q})$.

M is compact, so there exist finitely many $p_1, \ldots, p_l \in M$ such that

- $\{U'_j = \psi_{p_j}(s^{-1}_{p_j}(0)) \mid j = 1, \ldots, l\}$ is an open cover of M.
- There exists a Γ_{p_j}-invariant neighborhood V_j of $\psi^{-1}_{p_j}(p_j)$ in V_{p_j} such that $V_j \subset\subset V_{p_j}$ and $\{U_j = \psi_{p_j}(s^{-1}_{p_j}(0) \cap V_j) \mid j = 1, \ldots, l\}$ is still an open cover of M.

Let $\beta_j : V_{p_j} \to [0, 1]$ be a smooth function with compact support such that $\beta_j \equiv 1$ on the closure of V_j. For any $\nu = (\nu_1, \ldots, \nu_l)$, where $\nu_i : V_{p_j} \to E_{p_j}$ is a small continuous section, not necessarily Γ_{p_j}-equivariant, we construct a section $\nu_p : \hat{V}_p \to E_p|_{\hat{V}_p}$ for each $p \in X$, where $\hat{V}_p$ is an Γ_p-invariant neighborhood of $\psi^{-1}_p(p)$.

Given $p \in M$, if $p \in U_j$, then there is a Γ_p-invariant neighborhood $V_{p_j p}$ of $\psi^{-1}_p(p)$ such that we have the following commutative diagram:

$$\begin{array}{ccc} E_p|_{V_{p_j p}} & \xrightarrow{\hat{\phi}_{p_j p}} & E_{p_j} \\ \downarrow & & \downarrow \\ V_{p_j p} & \xrightarrow{\phi_{p_j p}} & V_{p_j} \end{array}$$

For any section $s : V_{p_j} \to E_{p_j}$, let $\phi^*_{p_j p} s : V_{p_j p} \to E_p|_{V_{p_j p}}$ be the unique section satisfying $\hat{\phi}_{p_j p} \circ \phi^*_{p_j p} s = s \circ \phi_{p_j p}$. Let $\hat{V}_p = \cap_{p \in U'_j} V_{p_j p}$. We define $\nu_p = \sum_{p \in U'_j} \phi^*_{p_j p}(\beta_j \nu_j) : \hat{V}_p \to E_p|_{\hat{V}_p}$.

There exist $\hat{p}_1, \ldots, \hat{p}_{\hat{l}} \in M$ such that $\{\hat{U}_j = \psi_{\hat{p}_j}(s^{-1}_{\hat{p}_j}(0) \cap \hat{V}_{\hat{p}_j}) \mid j = 1, \ldots, \hat{l}\}$ is an open cover of M. We may choose $\nu = (\nu_1, \ldots, \nu_l)$ such that $s_{\hat{p}_j} + \nu_{\hat{p}_j} : \hat{V}_{\hat{p}_j} \to E_{\hat{p}_j}|_{\hat{V}_{\hat{p}_j}}$ is smooth and intersects the zero section transversally for $j = 1, \ldots, \hat{l}$. So $\hat{M}^\nu_j = (s_{\hat{p}_j} + \nu_{\hat{p}_j})^{-1}(0)$ is a d-dimensional submanifold of $\hat{V}_{\hat{p}_j}$, where d is the virtual dimension of the Kuranishi structure. The orientation of the Kuranishi structure induces an orientation on $\hat{M}^\nu_j$. If $\hat{V}_{\hat{p}_j}$ has corners, we may further require that $\hat{M}^\nu_j$ intersect all the corners of $\hat{V}_{\hat{p}_j}$ transversally, i.e., $\hat{M}^\nu_j$ is a *neat* submanifold of $\hat{V}_{\hat{p}_j}$ in the sense of [17, Chapter 1, Section 4]. We call such $\nu = \{\nu_p : \hat{V}_p \to E_p|_{\hat{V}_p} \mid p \in M\}$ a *generic* perturbation. Note that the difference between two generic perturbations is smooth.

From our choice of $\nu_{\hat{p}_j}$, we have

$$\psi_{\hat{p}_j}(\hat{M}^\nu_j) \cap \psi_{\hat{p}_{j'}}(\hat{V}_{\hat{p}_{j'}}) = \psi_{\hat{p}_j}(\hat{V}_{p_j}) \cap \psi_{\hat{p}_{j'}}(\hat{M}^\nu_{j'}). \tag{7.1}$$

for any $j, j' \in \{1, \ldots, \hat{l}\}$.

Choose a triangulation of $\hat{M}^\nu_j$ so that it becomes a simplicial complex, and all its corners are subcomplexes. By (7.1), we may assume that there is a compact subcomplex K_j of $\hat{M}^\nu_j$ such that

- $\cup^{\hat{l}}_{j=1} \psi_{\hat{p}_j}(K_j) = \cup^{\hat{l}}_{j=1} \psi_{\hat{p}_j}(\hat{M}^\nu_j)$.

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- For any $j, j' \in \{1, \dots, \hat{l}\}$, $\psi_{\hat{p}_j}^{-1}\left(\psi_{\hat{p}_j}(K_j) \cap \psi_{\hat{p}_{j'}}(K_{j'})\right)$ is a subcomplex of the $(d-1)$-dimensional simplicial complex ∂K_j.

Let $\{\Delta^d_{j,\alpha} \mid \alpha = 1, \dots, N_j\}$ be the set of d-simplices in the triangulation of K_j. Each $\Delta^d_{j,\alpha}$ has an orientation induced by that of $\hat{M}^\nu_j$. The inclusion $i_{j,\alpha} : \Delta^d_{j,\alpha} \to K_j \subset \hat{V}_{\hat{p}_j}$ can be viewed as a singular d-chain in $\hat{V}_{\hat{p}_j}$. We define

$$\begin{aligned} K^\nu_j &= \sum_{\alpha=1}^{N_j} i_{j,\alpha} \in \mathcal{S}_d(\hat{V}_{\hat{p}_j}; \mathbb{Z}) \subset \mathcal{S}_d(\hat{V}_{\hat{p}_j}; \mathbb{Q}) \\ M_{\mathcal{K},\nu} &= \sum_{j=1}^{\hat{l}} \frac{1}{|\Gamma_{\hat{p}_j}|} \psi_{\hat{p}_j *} K^\nu_j \in \mathcal{S}_d(W; \mathbb{Q}) \end{aligned}$$

where $|\Gamma_{\hat{p}_j}|$ is the cardinality of $\Gamma_{\hat{p}_j}$. It follows from our construction that $\partial M_{\mathcal{K},\nu} \in \mathcal{S}_{d-1}(\partial W) \subset \mathcal{S}_{d-1}(W)$, so $M_{\mathcal{K},\nu}$ represents a singular relative d-cycle $\bar{M}_{\mathcal{K},\nu} \in \mathcal{S}_{d-1}(W, \partial W; \mathbb{Q})$, which represents a class $[M_{\mathcal{K},\nu}]^{\mathrm{rel}} \in H_d(W, \partial W; \mathbb{Q})$.

Note that up to subdivision, $M_{\mathcal{K},\nu} \in \mathcal{S}_d(W; \mathbb{Q})$ depends on $\nu = \{\nu_p : \hat{V}_p \to E_p|_{\hat{V}_p} \mid p \in M\}$ but not on the choice of $\hat{p}_1, \dots, \hat{p}_{\hat{l}} \in M$.

Proposition 7.4. *The class $[M_{\mathcal{K},\nu}]^{\mathrm{rel}} \in H_d(W, \partial W; \mathbb{Q})$ is independent of the choice of $\nu = \{\nu_p : \hat{V}_p \to E_p|_{\hat{V}_p} \mid p \in M\}$. So we may write $[M_{\mathcal{K}}]^{\mathrm{rel}}$ for this class.*

Proof. We first observe that a Kuranishi structure with corners $\mathcal{K}$ on M with ambient space W gives rise to a Kuranishi structure with corners $\mathcal{K} \times [0,1]$ on $M \times [0,1]$ with ambient space $W \times [0,1]$. To see this, consider $(p,t) \in M \times [0,1]$. Let $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ be the Kuranishi neighborhood of p assigned by the Kuranishi structure on M. Let $V_{(p,t)} = V_p \times [0,1]$, and let $\pi_p : V_{(p,t)} \to V_p$ be the projection to the first factor. Let $E_{(p,t)} = \pi_p^* E_p \to V_{(p,t)}$, and let $s_{(p,t)} = \pi_p^* s_p : V_{(p,t)} \to E_{(p,t)}$. We define $\psi_{(p,t)} = \psi_p \times id : V_{(p,t)} = V_p \times [0,1] \to W \times [0,1]$, where id is the identity map on $[0,1]$. Finally, let Γ_p act on $[0,1]$ trivially. Then $(V_{(p,t)}, E_{(p,t)}, \Gamma_p, \psi_{(p,t)}, s_{(p,t)})$ is a Kuranishi neighborhood of (p,t). The transition functions can be constructed from those of the Kuranishi structure on M in an obvious way.

Let $\nu = \{\nu_p : \hat{V}_p \to E_p|_{\hat{V}_p} \mid p \in M\}$, $\nu' = \{\nu'_p : \hat{V}_p \to E_p|_{\hat{V}_p} \mid p \in M\}$ be two choices of small generic perturbation of $\mathcal{K}$ in the above construction. There exists a generic perturbation

$$\mu = \{\mu_{(p,t)} : V_{(p,t)} \to E_{(p,t)} \mid (p,t) \in M \times [0,1]\}$$

of $\mathcal{K} \times [0,1]$ such that

$$i^*_{p,0} \mu_{(p,\frac{1}{2})} = \nu_p : V_p \to E_p,\ i^*_{p,1} \mu'_{(p,\frac{1}{2})} = \nu'_p : V_p \to E_p,$$

where $i_{p,t} : V_p \to V_{(p,\frac{1}{2})} = V_p \times [0,1]$ is the inclusion $x \mapsto (x,t)$.

From the paragraph right before Proposition 7.4, we may use $\hat{p}_1, \dots, \hat{p}_{\hat{l}}$ in the construction of both M^ν and $M^{\nu'}$. Let

$$Y_j = (s_{(\hat{p}_j,\frac{1}{2})} + \mu_{(\hat{p}_j,\frac{1}{2})})^{-1}(0) \subset \hat{V}_{\hat{p}_j} \times [0,1].$$

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Then

$$Y_j \cap \partial(\hat{V}_{\hat{p}_j} \times [0,1]) = i_{\hat{p}_j,1}(\hat{M}_j^{\nu'}) \cup i_{\hat{p}_j,0}(\hat{M}_j^{\nu}) \cup \Big(Y_j \cap (\partial\hat{V}_{\hat{p}_j} \times [0,1])\Big),$$

where

$$\partial(\hat{V}_{\hat{p}_j} \times [0,1]) = i_{\hat{p}_j,1}(\hat{V}_{\hat{p}_j}) \cup i_{\hat{p}_j,0}(\hat{V}_{\hat{p}_j}) \cup (\partial\hat{V}_{\hat{p}_j} \times [0,1])$$

is the union of corners of $\hat{V}_{\hat{p}_j} \times [0,1]$. This gives rise to a singular $(d+1)$-chain B_j in $\hat{V}_{\hat{p}_j} \times [0,1]$ such that

$$\partial B_j = (i_{\hat{p}_j,1})_* K_j^{\nu'} - (i_{\hat{p}_j,0})_* K_j^{\nu} + C_j + D_j,$$

where $(i_{\hat{p}_j,1})_* K_j^{\nu'}$ comes from $i_{\hat{p}_j,1}(\hat{M}_j^{\nu'})$, $(i_{\hat{p}_j,0})_* K_j^{\nu}$ comes from $i_{\hat{p}_j,0}(\hat{M}_j^{\nu})$, D_j comes from $Y_j \cap (\partial\hat{V}_{\hat{p}_j} \times [0,1])$, and C_j will get cancelled,

$$\sum_{j=1}^{\hat{l}} (\psi_{\hat{p}_j} \times id)_* C_j = 0 \in \mathcal{S}_d(W \times [0,1]; \mathbb{Q}).$$

Let $\pi_j : \hat{V}_{\hat{p}_j} \times [0,1] \to \hat{V}_{\hat{p}_j}$ be the projection to the first factor. We have

$$\partial({\pi_j}_* B_j) = K_j^{\nu'} - K_j^{\nu} + (\pi_j)_* C_j + (\pi_j)_* D_j.$$

Let

$$B = \sum_{j=1}^{\hat{l}} \frac{1}{|\Gamma_{\hat{p}_j}|} {\psi_{\hat{p}_j}}_* {\pi_j}_* B_j \in \mathcal{S}_{d+1}(W; \mathbb{Q}),$$

and

$$D = \sum_{j=1}^{\hat{l}} \frac{1}{|\Gamma_{\hat{p}_j}|} {\psi_{\hat{p}_j}}_* {\pi_j}_* D_j \in \mathcal{S}_d(\partial W; \mathbb{Q}) \subset \mathcal{S}_d(W; \mathbb{Q}).$$

Then

$$\partial B = M_{\mathcal{K},\nu'} - M_{\mathcal{K},\nu} + D \in \mathcal{S}_d(W; \mathbb{Q})$$

since

$$\sum_{j=1}^{\hat{l}} \frac{1}{|\Gamma_{\hat{p}_j}|} {\psi_{\hat{p}_j}}_* {\pi_j}_* C_j = \pi_* \left(\sum_{j=1}^{\hat{l}} \frac{1}{|\Gamma_{\hat{p}_j}|} (\psi_{\hat{p}_j} \times id)_* C_j \right) = 0,$$

where $\pi : W \times [0,1] \to W$ is the projection to the first factor. We have $\partial \bar{B} = \bar{M}_{\mathcal{K},\nu'} - \bar{M}_{\mathcal{K},\nu} \in \mathcal{S}_d(W, \partial W; \mathbb{Q})$, therefore

$$[M_{\mathcal{K},\nu'}]^{\mathrm{rel}} = [M_{\mathcal{K},\nu}]^{\mathrm{rel}} \in H_d(W, \partial W; \mathbb{Q}).$$

$\square$

Proposition 7.5. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two equivalent Kuranishi structures with corners on a compact Hausdorff topological space M. Let W be an ambient space of both $\mathcal{K}_1$ and $\mathcal{K}_2$. Then $[M_{\mathcal{K}_1}]^{\mathrm{rel}} = [M_{\mathcal{K}_2}]^{\mathrm{rel}}$.*

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Proof. Let

$$\begin{aligned}
\mathcal{K}_1 &= \Big\{(V_{1,p}, E_{1,p}, \Gamma_{1,p}, \psi_{1,p}, s_{1,p}) : p \in M, (V_{1,pq}, \hat{\phi}_{1,pq}, \phi_{1,pq}, h_{1,pq}) : q \in \psi_{1,p}(s_{1,p}^{-1}(0))\Big\} \\
\mathcal{K}_2 &= \Big\{(V_{2,p}, E_{2,p}, \Gamma_{2,p}, \psi_{2,p}, s_{2,p}) : p \in M, (V_{2,pq}, \hat{\phi}_{2,pq}, \phi_{2,pq}, h_{2,pq}) : q \in \psi_{2,p}(s_{2,p}^{-1}(0))\Big\} \\
\mathcal{K} &= \Big\{(V_p, E_p, \Gamma_p, \psi_p, s_p) : p \in M, (V_{pq}, \hat{\phi}_{pq}, \phi_{pq}, h_{pq}) : q \in \psi_p(s_p^{-1}(0))\Big\}
\end{aligned}$$

be as in Definition 6.5. Let $\nu_i = \{\nu_{i,p} : V_{i,p} \to E_{i,p} \mid p \in M\}$ be a generic perturbation which can be used to define a virtual fundamental chain $M_{\mathcal{K}_i,\nu_i} \in \mathcal{S}_d(W;\mathbb{Q})$, for $i = 1, 2$. ν_i can be extended to a generic perturbation $\mu_i = \{\mu_{i,p} : V_p \to E_p \mid p \in M\}$ such that $\hat{\phi}_i \circ \nu_{i,p} = \mu_{i,p} \circ \phi_i$. We have $M_{\mathcal{K}_i,\nu_i} = M_{\mathcal{K},\mu_i} \in \mathcal{S}_d(W;\mathbb{Q})$. Thus

$$[M_{\mathcal{K}_1}]^{\mathrm{rel}} = [M_{\mathcal{K},\mu_1}]^{\mathrm{rel}} = [M_{\mathcal{K},\mu_2}]^{\mathrm{rel}} = [M_{\mathcal{K}_2}]^{\mathrm{rel}} \in H_d(W, \partial W;\mathbb{Q}).$$

□

Remark 7.6. *Let M be a compact Hausdorff topological space with an oriented Kuranishi structure $\mathcal{K}$, and let W be an ambient space of $\mathcal{K}$. Then the above construction yields $M_{\mathcal{K},\nu} \in \mathcal{S}_d(W,\mathbb{Q})$ such that $\partial M_{\mathcal{K},\nu} = 0 \in \mathcal{S}_{d-1}(W,\mathbb{Q})$, so it represents a class $[M_{\mathcal{K},\nu}] \in H_d(W;\mathbb{Q})$. The proof of Proposition 7.4 shows that this class is independent of the choice of ν, so we may write $[M_{\mathcal{K}}]$ for this class. The proof of Proposition 7.5 shows that if $\mathcal{K}'$ is another Kuranishi structure on M such that $\mathcal{K}' \sim \mathcal{K}$ and W is also an ambient space of $\mathcal{K}'$, then $[M_{\mathcal{K}}] = [M_{\mathcal{K}'}] \in H_d(W;\mathbb{Q})$. Let $i : M \to W$ be the inclusion. Then $[M_{\mathcal{K}}] = i_*[M] \in H_d(W;\mathbb{Q})$, where $i_*[M]$ is defined in* [9, Section 6].

Example 7.7. *Let X be a Calabi-Yau 3-fold, and let $\beta \in H_2(X;\mathbb{Z})$. Let $\overline{M}_{g,0}(X,\beta)$ denote the moduli space of stable maps f from a genus-g prestable curve C to X such that $f_*[C] = \beta$. Then $\overline{M}_{g,0}(X,\beta)$ has an oriented Kuranishi structure $\mathcal{K}$* [9]. *The virtual dimension of $\mathcal{K}$ is 0 for any $g \geq 0$ and $\beta \in H_2(X;\mathbb{Z})$. This Kuranishi structure has an ambient space $W^{1,p}_{g,0}(X,\beta)$, the moduli space of stable $W^{1,p}$ maps from a genus g prestable curve C to X such that $f_*[C] = \beta$. Then $[\overline{M}_{g,0}(X,\beta)_{\mathcal{K}}] \in H_0(W^{1,p}_{g,0}(X,\beta);\mathbb{Q})$, and* $\deg[\overline{M}_{g,0}(X,\beta)_{\mathcal{K}}] \in \mathbb{Q}$ *is some Gromov-Witten invariant* [9, Section 17].

Example 7.8. *Let X be a Calabi-Yau 3-fold, L be a special Lagrangian submanifold. $\overline{M}_{(g,h),(n,\vec{m})}(X, L \mid \beta, \vec{\gamma}, \mu)$ is empty for $\mu \neq 0$. The tangent bundle of L is trivial, so L is spin. We have constructed an orientable Kuranishi structure with corners $\mathcal{K}$ on $\mathcal{M} = \overline{M}_{(g,h),(0,\vec{0})}(X, L \mid \beta, \vec{\gamma}, 0)$. The equivalence class of $\mathcal{K}$ is independent of choices in our construction. The virtual dimension of $\mathcal{K}$ is 0 for all $g, h, \beta, \vec{\gamma}$. The Kuranishi structure has an ambient space $\mathcal{W} = W^{1,p}_{(g,h),(0,\vec{0})}(X, L \mid \beta, \vec{\gamma}, 0)$. $[\mathcal{M}_{\mathcal{K}}]^{\mathrm{rel}} \in H_0(\mathcal{W}, \partial\mathcal{W};\mathbb{Q}) = 0$, so we cannot get a nontrivial number from $[\mathcal{M}_{\mathcal{K}}]^{\mathrm{rel}}$.*

Most enumerative predictions about holomorphic curves with Lagrangian boundary conditions concern the special case in Example 7.8. Our goal is to give a rigorous mathematical definition of these highly nontrivial enumerative numbers. From Example 7.8 we know that the relative cycle $[M_{\mathcal{K}}]^{\mathrm{rel}} \in H_d(W, \partial W;\mathbb{Q})$ is not the right object for our purpose. We want to remember the virtual fundamental *chain* $M_{\mathcal{K},\nu} \in \mathcal{S}_d(W,\mathbb{Q})$ which contains more information. For example, when the virtual

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dimension $d=0$, the 0-chain $M_{\mathcal{K},\nu}$ represents a class $[M_{\mathcal{K},\nu}]\in H_0(W;\mathbb{Q})$ since any 0-chain is a cycle. Let ν,ν' be two perturbations. From the proof of Proposition 7.5,

$$\partial B = M_{\mathcal{K},\nu} - M_{\mathcal{K},\nu'} + D,$$

where $B\in\mathcal{S}_1(W,\mathbb{Q})$, $D\in\mathcal{S}_0(\partial W,\mathbb{Q})$. Thus $[M_{\mathcal{K},\nu}]$ *depends on* ν. We want to impose extra constraint on ν such that if ν and ν' both satisfy the constraint then the above D is zero. In particular, if ν and ν' satisfy the same boundary conditions in the sense that $\nu_{\hat{p}_j}=\nu'_{\hat{p}_j}$ on $\partial\hat{V}_{\hat{p}_j}$ for all $j=1,\ldots,\hat{l}$, then $D=0$, so $[M_{\mathcal{K},\nu}]=[M_{\mathcal{K},\nu'}]\in H_0(W;\mathbb{Q})$, and $\deg[M_{\mathcal{K},\nu}]=\deg[M_{\mathcal{K},\nu'}]\in\mathbb{Q}$.

7.2. S^1-action.

Definition 7.9. *Let M be a Hausdorff topological space, and let $\hat{\varrho}: S^1\times M\to M$ be a continuous S^1 action. A Kuranishi structure with corners*

$$\mathcal{K}=\left\{(V_p,E_p,\Gamma_p,\psi_p,s_p): p\in M, (V_{pq},\hat{\phi}_{pq},\phi_{pq},h_{pq}): q\in\psi_p(s_p^{-1}(0))\right\}.$$

on M is $\hat{\varrho}$-equivariant on the boundary *if the Kuranishi neighborhood $(V_p,E_p,\Gamma_p,\psi_p,s_p)$ of any $p\in\partial M$ is $\hat{\varrho}$-equivariant in the sense that*

1. *There is a free continuous S^1-action on V_p which commutes with the action of Γ_p and leaves ∂V_p invariant.*
2. *$E_p\to V_p$ is an S^1-equivariant vector bundle.*
3. *$s_p: V_p\to E_p$ is an S^1-equivariant section.*
4. *$\psi_p: s_p^{-1}(0)\to M$ is S^1-equivariant, where S^1 acts on M by $\hat{\varrho}$.*

Remark 7.10. *Let $\mathcal{K}$ be as above, and let $\partial\mathcal{K}$ be the Kuranishi structure with corners on ∂M defined in Remark 6.4. Then the quotients of the Kuranishi neighborhoods and transition functions of $\partial\mathcal{K}$ by the free S^1 action define a Kuranishi structure $\partial\mathcal{K}/S^1$ of virtual dimension $d-2$ on $\partial M/S^1$.*

Remark 7.11. *Let M be a Hausdorff topological space with a continuous S^1-action $\hat{\varrho}: S^1\times M\to M$. Then there exists an $\hat{\varrho}$-equivariant Kuranishi structure with corners only if the action leaves ∂M invariant and has no fixed point on ∂M.*

Definition 7.12. *Let M be a Hausdorff topological space with a continuous S^1-action $\hat{\varrho}: S^1\times M\to M$. Let*

$$\mathcal{K}_1=\left\{(V_{1,p},E_{1,p},\Gamma_{1,p},\psi_{1,p},s_{1,p}): p\in M, (V_{1,pq},\hat{\phi}_{1,pq},\phi_{1,pq},h_{1,pq}): q\in\psi_{1,p}(s_{1,p}^{-1}(0))\right\}$$

and

$$\mathcal{K}_2=\left\{(V_{2,p},E_{2,p},\Gamma_{2,p},\psi_{2,p},s_{2,p}): p\in M, (V_{2,pq},\hat{\phi}_{2,pq},\phi_{2,pq},h_{2,pq}): q\in\psi_{2,p}(s_{2,p}^{-1}(0))\right\}$$

be two Kuranishi structures with corners on M which $\hat{\varrho}$-equivariant on the boundary. $\mathcal{K}_1$ and $\mathcal{K}_2$ are are $\hat{\varrho}$-equivalent *if*

- *There is another Kuranishi structure*

$$\mathcal{K}=\left\{(V_p,E_p,\Gamma_p,\psi_p,s_p): p\in M, (V_{pq},\hat{\phi}_{pq},\phi_{pq},h_{pq}): q\in\psi_p(s_p^{-1}(0))\right\}$$

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on M which is ϱ-equivariant on the boundary such that for all $p \in M$, $(V_{1,p}, E_{1,p}, \Gamma_{1,p}, \psi_{1,p}, s_{1,p})$, $(V_{2,p}, E_{2,p}, \Gamma_{2,p}, \psi_{2,p}, s_{2,p})$, and $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ satisfy the relation described in Definition 6.2.

- *The ϕ_i and $\hat{\phi}_i$ in Definition 6.2 are S^1-equivariant.*

In this case, we write $\mathcal{K}_1 \stackrel{\hat{\varrho}}{\sim} \mathcal{K}_2$.

Note that $\mathcal{K}_1 \stackrel{\hat{\varrho}}{\sim} \mathcal{K}_2 \Rightarrow \mathcal{K}_1 \sim \mathcal{K}_2$.

Definition 7.13. *Let M be a Hausdorff topological space with a continuous S^1-action $\hat{\varrho} : S^1 \times M \to M$. Let*

$$\mathcal{K} = \Big\{(V_p, E_p, \Gamma_p, \psi_p, s_p) : p \in M, (V_{pq}, \hat{\phi}_{pq}, \phi_{pq}, h_{pq}) : q \in \psi_p(s_p^{-1}(0))\Big\}$$

*be a Kuranishi structure with corners on M which is $\hat{\varrho}$-equivariant on the boundary. An ambient space W of $\mathcal{K}$ is $\hat{\varrho}$-*equivariant *if $\hat{\rho}$ extends to an action $S^1 \times W \to W$, and $\psi_p : V_p \to W$ is S^1-equivariant.*

We now assume that M is a *compact*, Hausdorff topological space with a continuous S^1-action $\hat{\varrho} : S^1 \times M \to M$. Suppose that $\mathcal{K}$ is an *oriented* Kuranishi structure with corners of virtual dimension *zero* on M. We further assume that $\mathcal{K}$ is $\hat{\varrho}$-equivariant on the boundary, and W is an $\hat{\varrho}$-equivariant ambient space of $\mathcal{K}$.

In this case, a generic perturbation $\nu = \{\nu_p : V_p \to E_p \mid p \in M\}$ of $\mathcal{K}$ is a perturbation such that $s_p + \nu_p$ intersects the zero section transversally at isolated points and is nowhere zero on ∂V_p. Let $\bar{p}$ denote the equivalent class of $p \in \partial M$ in $\partial M/S^1$. The virtual dimension of

$$\partial\mathcal{K}/S^1 = \Big\{((\bar{V}_{\bar{p}}, \bar{E}_{\bar{p}}, \Gamma_p, \bar{\psi}_{\bar{p}}, \bar{s}_{\bar{p}} : \bar{p} \in \partial M/S^1,\ (\bar{V}_{\bar{p}\bar{q}}, \bar{\phi}_{\bar{p}\bar{q}}, \bar{\hat{\phi}}_{\bar{p}\bar{q}}, \bar{h}_{\bar{p}\bar{q}}) : \bar{q} \in \bar{\psi}_{\bar{p}}(\bar{s}_{\bar{p}}^{-1}(0))\Big\}$$

is -2, so a generic perturbation $\bar{\nu} = \{\bar{\nu}_{\bar{p}} : \bar{V}_{\bar{p}} \to \bar{E}_{\bar{p}} \mid p \in \partial M/S^1\}$ of $\partial\mathcal{K}/S^1$ is a perturbation such that $\bar{s}_{\bar{p}} + \bar{\nu}_{\bar{p}}$ is nowhere zero for all $\bar{p} \in \partial M/S^1$. Let $Q_p : \partial V_p \to \bar{V}_{\bar{p}} = \partial V_p/S^1$ be the natural projection. Then $Q^*\bar{\nu} = \{(Q^*\bar{\nu})_p = Q^*\bar{\nu}_{\bar{p}} : \partial V_p \to E|_{\partial V_p} \mid p \in \partial M\}$ is a generic perturbation of $\partial\mathcal{K}$. We call such a perturbation an $\hat{\varrho}$-equivariant perturbation. A generic perturbation of $\partial\mathcal{K}$ can always be extended to a generic perturbation of $\mathcal{K}$, so there exists a generic perturbation of $\mathcal{K}$ whose restriction to $\partial\mathcal{K}$ is $\hat{\varrho}$-equivariant.

Proposition 7.14. *Let M be a compact Hausdorff topological space with a continuous S^1-action $\hat{\varrho} : S^1 \times M \to M$. Let $\mathcal{K}$ be a Kuranishi structure with corners of virtual dimension 0 on M which is $\hat{\varrho}$-equivariant on the boundary. Let W be a $\hat{\varrho}$-equivariant ambient space of $\mathcal{K}$. Let $\nu = \{\nu_p : V_p \to E_p \mid p \in M\}$ be a generic perturbation such that $\nu|_{\partial\mathcal{K}} = \{\nu_p|_{\partial V_p} : \partial V_p \to E_p|_{\partial V_p} \mid p \in \partial M\}$ is $\hat{\varrho}$-equivariant. Then $[M_{\mathcal{K},\nu}] \in H_0(W;\mathbb{Q})$ does not depend on the perturbation ν, so we may write $[M_{\mathcal{K}}^{\hat{\varrho}}]$ for this class. If $\mathcal{K}' \stackrel{\hat{\varrho}}{\sim} \mathcal{K}$, and W is also a $\hat{\varrho}$-equivariant ambient space of $\mathcal{K}'$, then $[M_{\mathcal{K}'}^{\hat{\varrho}}] = [M_{\mathcal{K}}^{\hat{\varrho}}] \in H_0(W;\mathbb{Q})$.*

Proof. Let

$$\nu = \{\nu_p : V_p \to E_p \mid p \in M\},\ \nu' = \{\nu'_p : V_p \to E_p \mid p \in M\}$$

be two generic perturbations of $\mathcal{K}$ such that $\nu|_{\partial\mathcal{K}} = Q^*\bar{\nu}$, $\nu'|_{\partial\mathcal{K}} = Q^*\bar{\nu}'$, where

$$\bar{\nu} = \{\bar{\nu}_{\bar{p}} : \bar{V}_{\bar{p}} \to \bar{E}_{\bar{p}} \mid \bar{p} \in \partial M/S^1\},\ \bar{\nu}' = \{\bar{\nu}'_{\bar{p}} : \bar{V}_{\bar{p}} \to \bar{E}_{\bar{p}} \mid \bar{p} \in \partial M/S^1\}$$

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are perturbations for $\partial\mathcal{K}/S^1$ such that $\bar{s}_{\bar{p}}+\bar{\nu}_{\bar{p}}$, $\bar{s}_{\bar{p}}+\bar{\nu}'_{\bar{p}}$ are nowhere zero for all $\bar{p}\in\partial M/S^1$. There exists a generic perturbation

$$\bar{\mu}=\{\bar{\mu}_{(\bar{p},t)}:V_{(\bar{p},t)}\to E_{(\bar{p},t)}\mid(\bar{p},t)\in\partial M/S^1\times[0,1]\}$$

such that

$$i^*_{\bar{p},0}\mu_{(\bar{p},\frac{1}{2})}=\bar{\nu}_p:\bar{V}_{\bar{p}}\to\bar{E}_{\bar{p}},\ i^*_{\bar{p},1}\mu_{(\bar{p},\frac{1}{2})}=\bar{\nu}'_p:\bar{V}_{\bar{p}}\to\bar{E}_{\bar{p}},$$

where $i_{\bar{p},t}:V_{\bar{p}}\to V_{(\bar{p},\frac{1}{2})}=V_{\bar{p}}\times[0,1]$ is the inclusion $x\mapsto(x,t)$. The virtual dimension of $\partial\mathcal{K}/S^1\times[0,1]$ is -1, so $\bar{\mu}$ being generic simply means that $\bar{s}_{(\bar{p},t)}+\bar{\mu}_{(\bar{p},t)}$ is nonzero for all $(\bar{p},t)\in\partial M/S^1\times[0,1]$. The pullback $Q^*\bar{\mu}$ of $\bar{\mu}$ is a generic perturbation of $\partial\mathcal{K}\times[0,1]$ such that $s_{(p,t)}+(Q^*\bar{\mu})_{(p,t)}$ is nowhere zero for any $(p,t)\in\partial M\times[0,1]$. We may extend $Q^*\bar{\mu}$ to a generic perturbation μ of $\mathcal{K}\times[0,1]$ such that

$$i^*_{p,0}\mu_{(p,\frac{1}{2})}=\nu_p:V_p\to E_p,\ i^*_{p,1}\mu_{(p,\frac{1}{2})}=\nu'_p:V_p\to E_p$$

as in the proof of Proposition 7.4. We proceed as the proof of Proposition 7.4, and use the notation there. Y_j does not intersect $\partial\hat{V}_{\hat{p}_j}\times[0,1]$, so $D=0$, and the 1-chain $B\in\mathcal{S}_1(W;\mathbb{Q})$ satisfies

$$\partial B=M_{\mathcal{K},\nu'}-M_{\mathcal{K},\nu}\in\mathcal{S}_0(W;\mathbb{Q}).$$

Therefore we have $[M_{\mathcal{K},\nu'}]=[M_{\mathcal{K},\nu}]\in H_0(W;\mathbb{Q})$.

The last statement can be proved as Proposition 7.5. □

7.3. **Invariants for an S^1-equivariant pair.** Let (X,ω) be a compact symplectic manifold together with an ω-tame almost complex structure J, and L be a Lagrangian submanifold. We assume that L is spin or that $h=1$ and L is relative spin so that $\mathcal{M}=\overline{M}_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)$ has an orientable Kuranishi structure. We fix an orientation by choosing a stable trivialization of TL or $TL\oplus V$ on the 2-skeleton $L^{(2)}$ of L, where V is chosen as in the proof of Theorem 6.36. $\mathcal{W}=W^{1,p}_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)$, is an ambient space of the Kuranishi structure on $\mathcal{M}$.

Definition 7.15. *An* admissible *S^1-action on (X,L) is an S^1-action $\varrho:S^1\times X\to X$ such that*

- *ϱ preserves J and L.*
- *The restriction of ϱ to L is* free.

We now assume that there is an admissible S^1-action on (X,L). Given $t=e^{i\theta}\in S^1$, let $f_t:X\to X$ be the J-holomorphic diffeomorphism given by $x\mapsto t\cdot x$. We have an S^1 action $\hat{\varrho}:S^1\times\mathcal{W}\to\mathcal{W}$ given by $(t,[(\lambda,u)])\mapsto t\cdot[(\lambda,u)]=[(\lambda,f_t\circ u)]$. The action preserves $\mathcal{M}\subset\mathcal{W}$ since f_t is J-holomorphic for all $t\in S^1$.

If $(\Sigma,\mathbf{B};\mathbf{p};\mathbf{q}^1,\dots,\mathbf{q}^h;u)$ represents a fixed point of the S^1-action, then $\Sigma=C\cup D_1\cup\cdots\cup D_h$, where C is a genus g prestable curve, and D_k is a disc which intersects C at an interior node for $k=1,\dots,h$. $u(\partial D_k)$ is an orbit of the S^1-action on L. Let $\mathcal{W}^{S^1}$ and $\mathcal{M}^{S^1}$ denote the fixed loci of the S^1-action on $\mathcal{W}$ and $\mathcal{M}$, respectively. Then $\mathcal{M}^{S^1}=\mathcal{W}^{S^1}\cap\mathcal{M}$, $\mathcal{W}^{S^1}\cap\partial\mathcal{W}=\emptyset$, and $\mathcal{M}^{S^1}\cap\partial\mathcal{M}=\emptyset$.

Theorem 7.16. *Let (X,L) be as above. Then there exists an oriented Kuranishi structure $\mathcal{K}$ on $\mathcal{M}=\overline{M}_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)$ which is ϱ-equivariant on the boundary. $\mathcal{W}=W^{1,p}_{(g,h),(n,\vec{m})}(X,L\mid\beta,\vec{\gamma},\mu)$ is a $\hat{\varrho}$-equivariant ambient space of $\mathcal{K}$.*

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Proof. Let

$$\mathcal{K} = \left\{(V'_\rho, E_\rho, \mathrm{Aut}\,\rho, \psi_\rho, s_\rho) : \rho \in \mathcal{M}, (V_{\rho\rho'}, \hat{\phi}_{\rho\rho'}, \phi_{\rho\rho'}, h_{\rho\rho'}), \rho' \in \psi_\rho(s^{-1}(0))\right\}$$

be a Kuranishi structure constructed as in Section 6. For $\rho \in \partial\mathcal{M}$, we will modify the Kuranishi neighborhood of ρ such that (1)-(4) in Definition 7.9 hold.

Given $\rho = (\lambda, u) \in \partial\mathcal{M}$, let $H_{\rho,\mathrm{domain}}$, $H_{\rho,\mathrm{map}} = \mathrm{Ker}(\pi \circ D_u)$ be defined as in Section 6. Recall that $\phi \in \mathrm{Aut}\,\lambda \mapsto u \circ \phi^{-1}$ induces an inclusion $H_{\rho,\mathrm{aut}} \subset H_{\rho,\mathrm{map}}$. Similarly, $t \in S^1 \mapsto f_t \circ u$ induces an inclusion $H_{\rho,\mathrm{circle}} = T_1 S^1 \subset H_{\rho,\mathrm{map}}$, where $T_1 S^1 \cong \mathbb{R}$ is the tangent line of $S^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ at $1 \in S^1$. Note that $H_{\rho,\mathrm{circle}} \subset H_{\rho,\mathrm{aut}}$ if and only if $\rho \in \mathcal{M}^{S^1}$, which is excluded since we consider $\rho \in \partial\mathcal{M}$. We may choose $H'_{\rho,\mathrm{map}}$ such that $H_{\rho,\mathrm{circle}} \subset H'_{\rho,\mathrm{map}}$ and $H_{\rho,\mathrm{map}} = H_{\rho,\mathrm{aut}} \oplus H'_{\rho,\mathrm{map}}$. Choose a subspace $H''_{\rho,\mathrm{map}}$ in $H'_{\rho,\mathrm{map}}$ such that $H'_{\rho,\mathrm{map}} = H_{\rho,\mathrm{circle}} \oplus H''_{\rho,\mathrm{map}}$.

Let $V''_{\rho,\mathrm{map}}$ be a small neighborhood of 0 in $H''_{\rho,\mathrm{map}}$, and $\epsilon > 0$ be small. Then for sufficiently small $(\xi, \eta, \eta') \in B_{\delta_2} \times D_d \times D'_{d'}$ and $w' \in H'_{\rho,\mathrm{map}}$, there are unique $w'' \in V''_{\rho,\mathrm{map}}$ and $\theta \in (-\epsilon, \epsilon)$ such that

$$u_{(\xi,\eta,\eta',w')} = f_{e^{i\theta}} \circ u_{(\xi,\eta,\eta',w'')} \equiv u_{(\xi,\eta,\eta',w'',e^{i\theta})},$$

where the notation is the same as in Section 6. Both $\mathrm{Aut}\,\rho$ and S^1 act on $B_{\delta_2} \times D_d \times D'_d \times V''_{\rho,\mathrm{map}} \times S^1$ such that $u_{\phi\cdot(\xi,\eta,\eta',w'',t)} = u_{(\xi,\eta,\eta',w'',t)} \circ \phi^{-1}$ and $t' \cdot (\xi, \eta, \eta', w'', t) = (\xi, \eta, \eta', w'', t't)$ for $\phi \in \mathrm{Aut}\,\rho$, $t' \in S^1$, $(\xi, \eta, \eta', w'', t) \in B_{\delta_2} \times D_d \times D'_d \times V''_{\rho,\mathrm{map}} \times S^1$. Note that the action of $\mathrm{Aut}\,\rho$ commutes with that of S^1. So we may choose a neighborhood V''_ρ of 0 in $B_{\delta_2} \times D_d \times D'_d \times V''_{\rho,\mathrm{map}}$ such that $V''_\rho \times S^1$ is invariant under the action of $\mathrm{Aut}\,\rho$. There is an S^1-equivariant map

$$\begin{aligned} \tilde{\psi}_\rho : V''_\rho \times S^1 &\longrightarrow \mathcal{W} \\ (\xi, \eta, \eta', w'', t) &\mapsto [(\lambda_{(\xi,\eta,\eta')}, u_{(\xi,\eta,\eta',w'',t)})] \end{aligned}$$

$(V''_\rho \times S^1)/\mathrm{Aut}\,\rho \to \mathcal{W}$ is injective if and only if $S^1_\rho = \{1\}$, where S^1_ρ is the stabilizer at ρ of the S^1-action. In general, S^1_ρ is a finite group because $\rho \in \partial M$ is not a fixed point of the S^1-action.

The automorphism of $\bar{\rho} \in \partial M/S^1$ is

$$\mathrm{Aut}\,\bar{\rho} = \{\phi \in \mathrm{Aut}\,\lambda \mid u \circ \phi = f_t \circ u \text{ for some } t \in S^1\},$$

and

$$S^1_\rho = \{t \in S^1 \mid u \circ \phi = f_t \circ u \text{ for some } \phi \in \mathrm{Aut}\,\lambda\}.$$

$\mathrm{Aut}\,\rho$ is a normal subgroup of $\mathrm{Aut}\,\bar{\rho}$, and we have an exact sequence

$$1 \to \mathrm{Aut}\,\rho \to \mathrm{Aut}\,\bar{\rho} \to S^1_\rho \to 1.$$

The action of $\mathrm{Aut}\,\rho$ on $V''_\rho \times S^1$ extends to $\mathrm{Aut}\,\bar{\rho}$, and $(V''_\rho \times S^1)/\mathrm{Aut}\,\bar{\rho} \to \mathcal{W}$ is injective.

Let $q_\rho : V''_\rho \times S^1 \to V''_\rho$ be the projection to the first factor, and let $E \to V''_\rho$ be the restriction of the obstruction bundle over V'_ρ. Let $\tilde{V}_\rho = V''_\rho \times S^1$, $\tilde{E}_\rho = q^*_\rho E \to \tilde{V}_\rho$, $\tilde{s}_\rho = q^*_\rho(s_\rho|_{V''_\rho})$. Then $(\tilde{V}_\rho, \tilde{E}_\rho, \mathrm{Aut}\,\bar{\rho}, \tilde{\psi}_\rho, \tilde{s}_\rho)$ is a Kuranishi neighborhood of $\rho \in \partial\mathcal{M}$ in $\mathcal{M}$ which satisfies (1)-(4) in Definition 7.9.

Finally, we proceed as in Section 6.5 to construct new transition functions. ☐

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The $\hat{\varrho}$-equivalence class of the Kuranishi structure with corners which is $\hat{\varrho}$-equivariant on the boundary constructed in the above proof does not depend on various choices.

Now consider the case $m = n = 0$, and the virtual dimension

$$d = \mu + (N-3)(2-2g-h) = 0.$$

We define the *Euler characteristic* of $\mathcal{M} = \overline{M}_{(g,h),(0,\vec{0})}(X, L \mid \beta, \vec{\gamma}, \mu)$ to be

$$\chi_{(g,h)}(X, L, \varrho|\beta, \vec{\gamma}, \mu) = \deg[\mathcal{M}^{\hat{\varrho}}_{\mathcal{K}}] \in \mathbb{Q},$$

where $\mathcal{K}$ is a Kuranishi structure with corners which is $\hat{\varrho}$-equivariant on the boundary, constructed in the proof of Theorem 7.16. This number is well-defined by Proposition 7.14.

$\chi_{(g,h)}(X, L, \varrho|\beta, \vec{\gamma}, \mu)$ is an invariant for the equivariant pair (X, L, ϱ), but *not* an invariant for the pair (X, L). In other words, it is possible that

$$\chi_{(g,h)}(X, L, \varrho_1|\beta, \vec{\gamma}, \mu) \neq \chi_{(g,h)}(X, L, \varrho_2|\beta, \vec{\gamma}, \mu)$$

for two different admissible S^1-actions ϱ_1, ϱ_2 on (X, L).

7.4. **Multiple covers of the disc.** We consider the special case studied in [24, 26, 33]. Let (z, u, v) and $(\tilde{z}, \tilde{u}, \tilde{v})$ be the two charts of $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$, related by $(\tilde{z}, \tilde{u}, \tilde{v}) = (\frac{1}{z}, zu, zv)$. Let X be the total space of $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$. There is an antiholomorphic involution

$$\begin{array}{rcl} A : X & \longrightarrow & X \\ (z, u, v) & \longmapsto & (\frac{1}{\bar{z}}, \bar{z}\bar{v}, \bar{z}\bar{u}) \end{array}$$

in terms of the first chart. The fixed locus $L = X^A$ is a special Lagrangian submanifold of the noncompact Calabi-Yau 3-fold. For any integer a, let $\varrho_a : S^1 \times X \to X$ be the S^1 action on X defined by $(e^{i\theta}, (z, u, v)) \mapsto (e^{i\theta} z, e^{i(a-1)\theta} u, e^{-ia\theta} v)$ on the first chart. Then ρ_a is an admissible S^1 action on (X, L). Let $D^2 = \{(z, 0, 0) \mid |z| \leq 1\}$ be a disc in the first chart, oriented by the complex structure. Then $\partial D^2 \subset L$. Let $\beta = [D^2] \in H_2(X, L; \mathbb{Z}) \cong H_2(\mathbf{P}^1, S^1; \mathbb{Z})$ and $\gamma = [\partial D^2] \subset H_1(L; \mathbb{Z}) \cong H_1(S^1; \mathbb{Z})$.

By Schwartz reflection principle (see e.g. [24, Section 3.3.2]), any nonconstant holomorphic map $f : (\Sigma, \partial\Sigma) \to (X, L)$ can be extended to a nonconstant holomorphic map $f_{\mathbb{C}} : \Sigma_{\mathbb{C}} \to X$, whose image must lie in $\mathbf{P}^1$. Thus we have

$$\mathcal{M} = \overline{M}_{(g,h),(0,\vec{0})}(X, L \mid d\beta, (n_1\gamma, \ldots, n_h\gamma), 0) = \overline{M}_{(g,h),(0,\vec{0})}(\mathbf{P}^1, S^1 \mid d\beta, (n_1\gamma, \ldots, n_h\gamma), 2d)$$

as topological spaces. Therefore, $\overline{M}_{(g,h),(0,\vec{0})}(X, L \mid d\beta, (n_1\gamma, \ldots, n_h\gamma), 0)$ is compact in the C^∞ topology. Note that the virtual dimension of the Kuranishi structure corners is 0 for $\overline{M}_{(g,h),(n,\vec{0})}(X, L \mid d\beta, (n_1\gamma, \ldots, n_h\gamma), 0)$ and $2(d+2g+h-2)$ for $\overline{M}_{(g,h),(n,\vec{0})}(\mathbf{P}^1, S^1 \mid d\beta, (n_1\gamma, \ldots, n_h\gamma), 0)$. In particular, these two Kuranishi structures on $\mathcal{M}$ are not equivalent. The following numbers are defined:

$$C(g, h|d; n_1, \ldots, n_h|a) = \chi_{(g,h)}(X, L, \varrho_a \mid d\beta, (n_1\gamma, \ldots, n_h\gamma), 0) \in \mathbb{Q},$$

where $n_1, \ldots, n_h$ are positive integers, and $d = n_1 + \cdots + n_h$.

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Let $\pi : \mathcal{C} \to \overline{M}_{g,h}$ be the universal family, ω_π be the relative dualizing sheaf, and $s_i : \overline{M}_{g,h} \to \mathcal{C}$ be the section corresponding to the i-th marked point. Let $\mathbb{E} = \pi_*\omega_\pi$ be the Hodge bundle on $\overline{M}_{g,h}$, and $\psi_i = c_1(s_i^*\omega_\pi)$.

Conclusion 7.17. *Let a be a positive integer. Then*

$$\begin{aligned}(-1)^{d-h}C(0;h|d;n_1,\ldots,n_h|a) &= C(0;h|d;n_1,\ldots,n_h|1-a)\\ &= (a(1-a))^{h-1}\prod_{i=1}^{h}\begin{pmatrix} n_i a-1\\ n_i-1\end{pmatrix} d^{h-3}.\end{aligned}$$

For $g>0$,

$$\begin{aligned}(-1)^{d-h}C(g;h|d;n_1,\ldots,n_h|a) &= C(g;h|d;n_1,\ldots,n_h|1-a)\\ &= (a(1-a))^{h-1}\prod_{i=1}^{h}\begin{pmatrix} n_i a-1\\ n_i-1\end{pmatrix}\cdot\\ &\int_{(\overline{M}_{g,h})_{U(1)}} \frac{c_g(\mathbb{E}^\vee(\lambda))c_g(\mathbb{E}^\vee((a-1)\lambda))c_g(\mathbb{E}^\vee(-a\lambda))\lambda^{2h-3}}{\prod_{i=1}^h(\lambda-n_i\psi_i)}.\end{aligned}$$

The above formulae for $C(g;h|d;n_1,\ldots,n_h|a)$ are calculated in [24] by localization techniques using the S^1-action ϱ_a. Actually, the definition of $\chi_{(g,h)}(X,L,\rho \mid \beta,\vec{\gamma},\mu)$ is inspired by R. Bott's interpretation of the computations in [24]. The localization formula, and in particular the proof of Conjecture 7.17, is left to future work.

Finally, the assumption of the existence of an admissible S^1-action is too restrictive. The S^1-action disappears when we perturb the almost complex structure J or the Lagrangian submanifold L, so the invariant is not even defined for other almost complex structures, and it is not clear in which sense $\chi_{(g,h)}(X,L,\rho \mid \beta,\vec{\gamma},\mu)$ is an "invariant". It is desirable to find a natural way to impose boundary conditions for the general case.

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